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Data: 15/04/2009
Semestre:
Curso: Engenharia de Computação
Disciplina: Álgebra Linear
Prova: I

1. *4 pts.* Dado as matrizes:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Encontrar: $\det \underline{\underline{A}}$ e $\det \underline{\underline{B}}$.

(b) Encontrar as matrizes adjuntas: $\underline{\underline{A}}^*$ e $\underline{\underline{B}}^*$.

(c) Encontrar os inversos: $\underline{\underline{A}}^{-1}$ e $\underline{\underline{B}}^{-1}$.

(d) Encontrar os determinantes dos inversos: $\det(\underline{\underline{A}}^{-1})$ e $\det(\underline{\underline{B}}^{-1})$.

(e) Encontrar os produtos: $\underline{\underline{A}}\,\underline{\underline{B}}$ e $\underline{\underline{B}}\,\underline{\underline{A}}$.

(f) Mostre que em geral vale por matrizes do mesmo ordem: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$.

(g) Encontrar o inverso do produto: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1}$.

(h) Encontrar o inverso do produto: $(\underline{\underline{B}}\,\underline{\underline{A}})^{-1}$.

Solution:

(a)

$$\det \underline{\underline{A}} = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

$$\det \underline{\underline{B}} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = (-1) \cdot 2 \cdot 3 = -6$$

(b) Para $\underline{\underline{A}}$:

$$D_{11} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \qquad D_{12} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad D_{13} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$$

$$D_{21} = \begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} = -2 \qquad D_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \qquad D_{23} = \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} = 0$$

$$D_{31} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = -2 \qquad D_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad D_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

Assim:

$$A_{11} = 3 \quad A_{12} = 1 \quad A_{13} = -1$$
$$A_{21} = 2 \quad A_{22} = 1 \quad A_{23} = 0$$
$$A_{31} = -2 \quad A_{32} = -1 \quad A_{33} = 1$$

Ou seja:

$$\underline{\underline{A}}^* = (A_{ji}) = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Por $\underline{\underline{B}}$ ser diagonal:

$$D_{ij} = A_{ij} = 0, \; i \neq j$$

Quem entende sério somente por sério
E brincadeira somente por brincadeira
De fato desentendeu ambos... *Piet Hein*

$$D_{11}=A_{11}=\begin{vmatrix}2&0\\0&3\end{vmatrix}=6\qquad D_{22}=A_{22}=\begin{vmatrix}-1&0\\0&3\end{vmatrix}=-3\qquad D_{33}=A_{33}=\begin{vmatrix}-1&0\\0&2\end{vmatrix}=-2$$

Ou seja:

$$\underline{\underline{B}}^*=\begin{pmatrix}6&0&0\\0&-3&0\\0&0&-2\end{pmatrix}$$

(c) Encontrar os inversos: $\underline{\underline{A}}^{-1}$ e $\underline{\underline{B}}^{-1}$.

$$\underline{\underline{A}}^{-1}=\frac{1}{\det\underline{\underline{A}}}\underline{\underline{A}}^*=\begin{pmatrix}3&2&-2\\1&1&-1\\-1&0&1\end{pmatrix}$$

E:

$$\underline{\underline{B}}^{-1}=\frac{1}{\det\underline{\underline{B}}}\underline{\underline{B}}^*=\begin{pmatrix}-1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}$$

(d)

$$\det(\underline{\underline{A}}^{-1})=\frac{1}{\det\underline{\underline{A}}}=1$$

$$\det(\underline{\underline{B}}^{-1})=\frac{1}{\det\underline{\underline{B}}}=-\frac{1}{6}$$

(e)

$$\underline{\underline{A}}\,\underline{\underline{B}}=\begin{pmatrix}1&-2&0\\0&1&1\\1&-2&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&2&0\\0&0&3\end{pmatrix}=\begin{pmatrix}1&-4&0\\0&2&3\\-1&-4&3\end{pmatrix}$$

$$\underline{\underline{B}}\,\underline{\underline{A}}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}1&-2&0\\0&1&1\\1&-2&1\end{pmatrix}=\begin{pmatrix}-1&2&0\\0&2&2\\3&-6&3\end{pmatrix}$$

(f) Por o inverso ser único e o produto de matrizes ser associativo:

$$(\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1})(\underline{\underline{A}}\,\underline{\underline{B}})=\underline{\underline{B}}^{-1}(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{\underline{B}}=\underline{\underline{B}}^{-1}\underline{\underline{I}}\,\underline{\underline{B}}=\underline{\underline{B}}^{-1}\underline{\underline{B}}=\underline{\underline{I}}$$

E:

$$(\underline{\underline{A}}\,\underline{\underline{B}})\,(\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1})=\underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{B}}^{-1})\underline{\underline{A}}^{-1}=\underline{\underline{A}}\,\underline{\underline{I}}\,\underline{\underline{A}}^{-1}=\underline{\underline{A}}\,\underline{\underline{A}}^{-1}=\underline{\underline{I}}$$

QED

(g)

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1}=\underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}=\begin{pmatrix}-1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}3&2&-2\\1&1&-1\\-1&0&1\end{pmatrix}=\begin{pmatrix}-3&-2&2\\\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{3}&0&\frac{1}{3}\end{pmatrix}$$

(h) Encontrar o inverso do produto: $(\underline{\underline{B}}\,\underline{\underline{A}})^{-1}$.

$$(\underline{\underline{B}}\,\underline{\underline{A}})^{-1}=\underline{\underline{A}}^{-1}\underline{\underline{B}}^{-1}=\begin{pmatrix}3&2&-2\\1&1&-1\\-1&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}-3&1&-\frac{2}{3}\\-1&\frac{1}{2}&-\frac{1}{3}\\1&0&\frac{1}{3}\end{pmatrix}$$

2. *4 pts.* Dado a sistema linear:

$$(*):\left\{\begin{array}{ccccccccccc}x_1&+&2x_2&+&3x_3&+&4x_4&+&5x_5&=&1\\2x_1&+&3x_2&+&4x_3&+&5x_4&-&x_5&=&2\\3x_1&+&4x_2&+&5x_3&+&6x_4&-&3x_5&=&3\end{array}\right\}$$

(a) Encontrar a solução completa do sistema homogêneo do $(*)$.

(b) Encontrar a solução completa do sistema não homogêneo.

(c) Encontrar a solução completa do sistema:

$$(**) : \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 3 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 4 \end{array} \right\}$$

Solution:
Resolvemos tudo de uma vez:

$$\left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \\ 2 & 3 & 4 & 5 & -1 & | & 2 & 3 \\ 3 & 4 & 5 & 6 & -3 & | & 3 & 4 \end{array} \right) \sim$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & | & 1 & 2 \\ 0 & -1 & -2 & -3 & -11 & | & 0 & -1 \\ 0 & -2 & -4 & -6 & -18 & | & 0 & -2 \end{array} \right) \sim$$

$$\left(\begin{array}{cccccccc} 1 & 0 & -1 & -2 & -17 & | & 1 & 0 \\ 0 & 1 & 2 & 3 & 11 & | & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 & | & 0 & 0 \end{array} \right) \sim$$

$$\left(\begin{array}{cccccccc} 1 & 0 & -1 & -2 & 0 & | & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & | & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 0 & 0 \end{array} \right)$$

Pondo $x_3 = t$ e $x_4 = s$ obtemos a solução completa do sistema homogêneo:

$$(x_1, x_2, x_3, x_4, x_5)^T = (t + 2s, -2t - 3s, t, s, 0)^T, \ \ (t, s) \in \mathbb{R}^2$$

E por $(*)$:

$$(x_1, x_2, x_3, x_4, x_5)^T = (1 + t + 2s, -2t - 3s, t, s, 0)^T, \ \ (t, s) \in \mathbb{R}^2$$

E por $(**)$:

$$(x_1, x_2, x_3, x_4, x_5)^T = (t + 2s, 1 - 2t - 3s, t, s, 0)^T, \ \ (t, s) \in \mathbb{R}^2$$

Comentário: Verificar:
(i) Verifique que a solução da eq. homogênea satisfaz as eqs.
(ii) Verifique que as soluções particulares satisfaz $(*)$ resp. $(**)$.

3. *2 pts.* (Cabeludo) Dado as matrizes:

$$\underline{\underline{A}} = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{array} \right), \quad (a, b) \in \mathbb{R}^2$$

$$\underline{\underline{B}} = \left(\begin{array}{cccc} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{array} \right), \quad a \in \mathbb{R}$$

(a) Encontrar para quaisquer valores de a e b o determinante do $\underline{\underline{A}}$.

(b) Encontrar para quaisquer valores de a e b o posto do $\underline{\underline{A}}$.

(c) Encontrar para qualquer valor de a o determinante do $\underline{\underline{B}}$.

(d) Para quais valores de a o matriz $\underline{\underline{B}}$ tem inverso? Para estes valores, encontrar o inverso.

Solution:

(a)

$$\det \underline{\underline{A}} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-1 \end{vmatrix} =$$

$$(-1)^{3+1}(-1)^{4+4}(a-1)(b-1)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (a-1)(b-1)(4-9) = -5(a-1)(b-1)$$

(b) Como no item anterior:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-1 \end{pmatrix}$$

Donde segue:

$$\rho_{\underline{\underline{A}}} = \begin{cases} 4, & a \neq 1 \wedge b \neq 1 \\ 3, & a = 1 \wedge b \neq 1 \\ 3, & a \neq 1 \wedge b = 1 \\ 2, & a = b = 1 \end{cases}$$

(c)

$$\det \underline{\underline{B}} = \begin{vmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{vmatrix} = a^4 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = a^4$$

(d) O inverso existe se e somente se o determinante não for zero: $a^4 \neq 0 \quad \Leftrightarrow \quad a \neq 0$.
Por $a \neq 0$ obtemos:

$$\left(\begin{array}{cccc|cccc} 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ a & 0 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & a & -1 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} a & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & -1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ 0 & 1 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 1 & -\frac{1}{a} & 0 & \frac{1}{a} & 0 \end{array}\right)$$

Assim:

$$\underline{\underline{B}}^{-1} = \begin{pmatrix} 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} \\ -\frac{1}{a} & 0 & \frac{1}{a} & 0 \end{pmatrix}$$

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Curso:	Engenharia Civil
Disciplina:	Álgebra Linear
Prova:	I

1. *4 pts.* Dado as matrizes:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Sabendo que:

$$\det(\underline{\underline{A}}^{-1}) = \det \underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}}$$

e:

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$$

responde o seguinte:

(a) Encontrar: $\det \underline{\underline{A}}$ e $\det \underline{\underline{B}}$.

(b) Encontrar as matrizes adjuntas: $\underline{\underline{A}}^*$ e $\underline{\underline{B}}^*$.

(c) Encontrar os inversos: $\underline{\underline{A}}^{-1}$ e $\underline{\underline{B}}^{-1}$.

(d) Encontrar os determinantes dos inversos: $\det(\underline{\underline{A}}^{-1})$ e $\det(\underline{\underline{B}}^{-1})$.

(e) Encontrar os produtos: $\underline{\underline{A}}\,\underline{\underline{B}}$ e $\underline{\underline{B}}\,\underline{\underline{A}}$.

(f) Encontrar o inverso do produto: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1}$.

Solution:
4 pts. Dado as matrizes:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Sabendo que:

$$\det(\underline{\underline{A}}^{-1}) = \det \underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}}$$

e:

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$$

responde o seguinte:

(a) Encontrar: $\det \underline{\underline{A}}$ e $\det \underline{\underline{B}}$.

$$\det \underline{\underline{A}} = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

$$\det \underline{\underline{B}} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = (-1) \cdot 2 \cdot 3 = -6$$

(b) Encontrar as matrizes adjuntas: $\underline{\underline{A}}^*$ e $\underline{\underline{B}}^*$.
Para $\underline{\underline{A}}$:

$$D_{11} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \qquad D_{12} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad D_{13} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$$

$$D_{21} = \begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} = -2 \qquad D_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \qquad D_{23} = \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} = 0$$

$$D_{31} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = -2 \qquad D_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad D_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$$

Assim:

$$A_{11} = 3 \quad A_{12} = 1 \quad A_{13} = -1$$
$$A_{21} = 2 \quad A_{22} = 1 \quad A_{23} = 0$$
$$A_{31} = -2 \quad A_{32} = -1 \quad A_{33} = 1$$

Ou seja:

$$\underline{\underline{A}}^* = (A_{ji}) = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Por $\underline{\underline{B}}$ ser diagonal:

$$D_{ij} = A_{ij} = 0,\ i \neq j$$

$$D_{11} = A_{11} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \qquad D_{22} = A_{22} = \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} = -3 \qquad D_{33} = A_{33} = \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

Ou seja:

$$\underline{\underline{B}}^* = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(c) Encontrar os inversos: $\underline{\underline{A}}^{-1}$ e $\underline{\underline{B}}^{-1}$.

$$\underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}} \underline{\underline{A}}^* = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

E:

$$\underline{\underline{B}}^{-1} = \frac{1}{\det \underline{\underline{B}}} \underline{\underline{B}}^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(d) Encontrar os determinantes dos inversos: $\det(\underline{\underline{A}}^{-1})$ e $\det(\underline{\underline{B}}^{-1})$.

$$\det(\underline{\underline{A}}^{-1}) = \frac{1}{\det \underline{\underline{A}}} = 1$$

$$\det(\underline{\underline{B}}^{-1}) = \frac{1}{\det \underline{\underline{B}}} = -\frac{1}{6}$$

(e) Encontrar os produtos: $\underline{\underline{A}}\,\underline{\underline{B}}$ e $\underline{\underline{B}}\,\underline{\underline{A}}$.

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 0 \\ 0 & 2 & 3 \\ -1 & -4 & 3 \end{pmatrix}$$

$$\underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & 2 \\ 3 & -6 & 3 \end{pmatrix}$$

(f) Encontrar o inverso do produto: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1}$.

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

2. *4 pts.* Dado a sistema linear:

$$(*) : \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 2 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 3 \end{array} \right\}$$

(a) Encontrar a solução completa do sistema homogêneo do $(*)$.

(b) Encontrar a solução completa do sistema não homogêneo.

(c) Encontrar a solução completa do sistema:

$$(**) : \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 3 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 4 \end{array} \right\}$$

Solution:

4 pts. Dado a sistema linear:

$$(*) : \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 2 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 3 \end{array} \right\}$$

(a) Encontrar a solução completa do sistema homogêneo do $(*)$.

(b) Encontrar a solução completa do sistema não homogêneo.

(c) Encontrar a solução completa do sistema:

$$(**) : \left\{ \begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 5x_5 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 3 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & - & 3x_5 & = & 4 \end{array} \right\}$$

Resolvemos tudo de uma vez:

$$\left(\begin{array}{ccccc|cc} 1 & 2 & 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & -1 & 2 & 3 \\ 3 & 4 & 5 & 6 & -3 & 3 & 4 \end{array} \right) \sim \left(\begin{array}{ccccc|cc} 1 & 2 & 3 & 4 & 5 & 1 & 2 \\ 0 & -1 & -2 & -3 & -11 & 0 & -1 \\ 0 & -2 & -4 & -6 & -18 & 0 & -2 \end{array} \right) \sim$$

$$\left(\begin{array}{ccccc|cc} 1 & 0 & -1 & -2 & -17 & 1 & 0 \\ 0 & 1 & 2 & 3 & 11 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccccc|cc} 1 & 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right)$$

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Pondo $x_3 = t$ e $x_4 = s$ obtemos a solução completa do sistema homogêneo:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t+2s \\ -2t-3s \\ t \\ s \\ 0 \end{pmatrix}, \quad (t,s) \in \mathbb{R}^2$$

E por $(*)$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+t+2s \\ -2t-3s \\ t \\ s \\ 0 \end{pmatrix}, \quad (t,s) \in \mathbb{R}^2$$

E por $(**)$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t+2s \\ 1-2t-3s \\ t \\ s \\ 0 \end{pmatrix}, \quad (t,s) \in \mathbb{R}^2$$

Comentário: Controle:
(i) Verifique que a solução da eqação homogênea satisfaz as eqções.
(ii) Verifique que as soluções particulares satisfaz $(*)$ resp. $(**)$.

3. *2 pts.* (Cabeludo) Dado as matrizes:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{pmatrix}, \quad (a,b) \in \mathbb{R}^2$$

$$\underline{\underline{B}} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad a \in \mathbb{R}$$

(a) Encontrar para quaisquer valores de a e b o determinante do $\underline{\underline{A}}$.
(b) Encontrar para quaisquer valores de a e b o posto do $\underline{\underline{A}}$.
(c) Encontrar para qualquer valor de a o determinante do $\underline{\underline{B}}$.
(d) Para quais valores de a o matriz $\underline{\underline{B}}$ tem inverso? Para estes valores, encontrar o inverso.

Solution:
2 pts. (Cabeludo) Dado as matrizes:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{pmatrix}, \quad (a,b) \in \mathbb{R}^2$$

$$\underline{\underline{B}} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad a \in \mathbb{R}$$

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(a) Encontrar para quaisquer valores de a e b o determinante do $\underline{\underline{A}}$.

$$\det \underline{\underline{A}} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-1 \end{vmatrix} =$$

$$(-1)^{3+1}(-1)^{4+4}(a-1)(b-1)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (a-1)(b-1)(4-9) = -5(a-1)(b-1)$$

(b) Encontrar para quaisquer valores de a e b o posto do $\underline{\underline{A}}$.
Como no item anterior:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a & 2 & 3 & 4 \\ 4 & 3 & 2 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ a-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-1 \end{pmatrix}$$

Donde segue:

$$\rho_{\underline{\underline{A}}} = \begin{cases} 4, & a \neq 1 \wedge b \neq 1 \\ 3, & a = 1 \wedge b \neq 1 \\ 3, & a \neq 1 \wedge b = 1 \\ 2, & a = b = 1 \end{cases}$$

(c) Encontrar para qualquer valor de a o determinante do $\underline{\underline{B}}$.

$$\det \underline{\underline{B}} = \begin{vmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ 0 & 0 & a & 0 \end{vmatrix} = a^4 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = a^4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = a^4$$

(d) Para quais valores de a o matriz $\underline{\underline{B}}$ tem inverso? Para estes valores, encontrar o inverso.
O inverso existe se e somente se o determinante não for zero: $a^4 \neq 0 \quad \Leftrightarrow \quad a \neq 0$.
Por $a \neq 0$ obtemos:

$$\left(\begin{array}{cccc|cccc} 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ a & 0 & a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & a & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & a & -1 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} a & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & -1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ 0 & 1 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 1 & -\frac{1}{a} & 0 & \frac{1}{a} & 0 \end{array}\right)$$

Assim:

$$\underline{\underline{B}}^{-1} = \begin{pmatrix} 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} \\ -\frac{1}{a} & 0 & \frac{1}{a} & 0 \end{pmatrix}$$

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Data:	27/05/2009
Semestre:	
Curso:	Engenharia de Computação
Disciplina:	Álgebra Linear
Prova:	II

1. *2 pts.* Dado os vetores em relação ao base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4, \underline{\mathbf{e}}_5)$, em $\mathbb{R}^5$:

$$\underline{\mathbf{v}}_1 = (1,1,1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,-1,1,-1,1)^T \qquad \underline{\mathbf{v}}_3 = (3,-1,3,-1,3)^T \qquad \underline{\mathbf{v}}_4 = (0,1,0,1,0)^T$$

   (a) Mostre que: $V = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4)$. Qual a dimensão do V?

   (b) Escreve $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$ como combinações lineares de $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$.

   (c) Escreve $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$ como combinações lineares de $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$.

Solution:

2 pts. Dado os vetores em relação ao base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4, \underline{\mathbf{e}}_5)$, em $\mathbb{R}^5$:

$$\underline{\mathbf{v}}_1 = (1,1,1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,-1,1,-1,1)^T \qquad \underline{\mathbf{v}}_3 = (3,-1,3,-1,3)^T \qquad \underline{\mathbf{v}}_4 = (0,1,0,1,0)^T$$

(a) Mostre que: $V = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4)$. Qual a dimensão do V?

(b) Escreve $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$ como combinações lineares de $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$.

(c) Escreve $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$ como combinações lineares de $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$.

<u>Solução:</u>

(a) Formamos o matriz com os vetores em colunas:

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Dai segue: $dim\, ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = dim\, ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = dim\, ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = 2$, o que implica: $ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = 2$. A dimensão é o posto do matriz $\underline{\underline{\mathbf{V}}}$, isto é 2.

(b) Omitindo as linhas com somente zeros:

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 0 \\ 0 & -2 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 2 & -\frac{1}{2} \end{array}\right)$$

Segue: $\underline{\mathbf{v}}_3 = \underline{\mathbf{v}}_1 + 2\underline{\mathbf{v}}_2$, e: $\underline{\mathbf{v}}_4 = \frac{1}{2}\underline{\mathbf{v}}_1 - \frac{1}{2}\underline{\mathbf{v}}_2$.

(c) Trocando o ordem dos vetores:

$$\left(\begin{array}{cc|cc} 3 & 0 & 1 & 1 \\ -4 & 1 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ -4 & 1 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{3} & -\frac{2}{3} \end{array}\right)$$

Segue: $\underline{\mathbf{v}}_1 = \frac{1}{3}\underline{\mathbf{v}}_3 + \frac{4}{3}\underline{\mathbf{v}}_4$, e: $\underline{\mathbf{v}}_2 = \frac{1}{3}\underline{\mathbf{v}}_3 - \frac{2}{3}\underline{\mathbf{v}}_4$.

2. *6 pts.* Dado os vetores em relação ao base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4)$, em $\mathbb{R}^4$:

$$\underline{\mathbf{v}}_1 = (1,1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,-1,1,-1)^T \qquad \underline{\mathbf{v}}_3 = (1,1,-1,-1)^T \qquad \underline{\mathbf{v}}_4 = (1,-1,-1,1)^T$$

   (a) Mostre que os $\underline{\mathbf{v}}_i$'s são mutualmente ortogonais, isto é: $\underline{\mathbf{v}}_i \cdot \underline{\mathbf{v}}_j = 0$ por $i \neq j$.

   (b) Encontrar um base *ortonormal* de $\mathbb{R}^4$, $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$, tal que: $\underline{\mathbf{d}}_i = c_i \underline{\mathbf{v}}_i$.

(c) Encontrar uma relação matricial entre os coordenados em relação aos $\underline{\mathbf{e}}_i$'s (antigos), $\underline{\mathbf{x}}_A$, e os coordenados em relação aos $\underline{\mathbf{d}}_i$'s (novos), $\underline{\mathbf{x}}_N$:

$$\underline{\mathbf{x}}_A = \underline{\underline{\mathbf{D}}}\,\underline{\mathbf{x}}_N$$

(d) Encontrar uma relação matricial entre os coordenados novos, $\underline{\mathbf{x}}_N$, e os coordenados antigos, $\underline{\mathbf{x}}_A$:

$$\underline{\mathbf{x}}_N = \underline{\underline{\mathbf{D}}}'\,\underline{\mathbf{x}}_A$$

(e) Justificar que vale: $\underline{\underline{\mathbf{D}}}' = \underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T = \underline{\underline{\mathbf{D}}}$.

(f) Encontrar os coordenados dos vetores:

$$\underline{\mathbf{w}}_1 = (-1, 1, -1, 1)^T \qquad \underline{\mathbf{w}}_2 = (1, 2, 3, 4)^T$$

em relação ao base novo, $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$.

Solution:

6 pts. Dado os vetores em relação ao base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4)$, em $\mathbb{R}^4$:

$$\underline{\mathbf{v}}_1 = (1, 1, 1, 1)^T \qquad \underline{\mathbf{v}}_2 = (1, -1, 1, -1)^T \qquad \underline{\mathbf{v}}_3 = (1, 1, -1, -1)^T \qquad \underline{\mathbf{v}}_4 = (1, -1, -1, 1)^T$$

(a) Mostre que os $\underline{\mathbf{v}}_i$'s são mutualmente ortogonais, isto é: $\underline{\mathbf{v}}_i \cdot \underline{\mathbf{v}}_j = 0$ por $i \neq j$.

(b) Encontrar um base *ortonormal* de $\mathbb{R}^4$, $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$, tal que: $\underline{\mathbf{d}}_i = c_i \underline{\mathbf{v}}_i$.

(c) Encontrar uma relação matricial entre os coordenados em relação aos $\underline{\mathbf{e}}_i$'s (antigos), $\underline{\mathbf{x}}_A$, e os coordenados em relação aos $\underline{\mathbf{d}}_i$'s (novos), $\underline{\mathbf{x}}_N$:

$$\underline{\mathbf{x}}_A = \underline{\underline{\mathbf{D}}}\,\underline{\mathbf{x}}_N$$

(d) Encontrar uma relação matricial entre os coordenados novos, $\underline{\mathbf{x}}_N$, e os coordenados antigos, $\underline{\mathbf{x}}_A$:

$$\underline{\mathbf{x}}_N = \underline{\underline{\mathbf{D}}}'\,\underline{\mathbf{x}}_A$$

(e) Justificar que vale: $\underline{\underline{\mathbf{D}}}' = \underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T = \underline{\underline{\mathbf{D}}}$.

(f) Encontrar os coordenados dos vetores:

$$\underline{\mathbf{w}}_1 = (-1, 1, -1, 1)^T \qquad \underline{\mathbf{w}}_2 = (1, 2, 3, 4)^T$$

em relação ao base novo, $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$.

Solução:

(a) Verificamos:

$$\underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 = 1 - 1 + 1 - 1 = 0 \qquad \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_3 = 1 + 1 - 1 - 1 = 0 \qquad \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_4 = 1 - 1 - 1 + 1 = 0$$
$$\underline{\mathbf{v}}_2 \cdot \underline{\mathbf{v}}_3 = 1 - 1 - 1 + 1 = 0 \qquad \underline{\mathbf{v}}_2 \cdot \underline{\mathbf{v}}_4 = 1 + 1 - 1 - 1 = 0$$
$$\underline{\mathbf{v}}_3 \cdot \underline{\mathbf{v}}_4 = 1 - 1 + 1 - 1 = 0$$

QED[1].

(b) Normalizamos os $\underline{\mathbf{v}}_i$'s: $|\underline{\mathbf{v}}_1| = |\underline{\mathbf{v}}_2| = |\underline{\mathbf{v}}_3| = |\underline{\mathbf{v}}_4| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$. Assim:

$$\underline{\mathbf{d}}_1 = \frac{\underline{\mathbf{v}}_1}{|\underline{\mathbf{v}}_1|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$$
$$\underline{\mathbf{d}}_2 = \frac{\underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_2|} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$$

[1] Quod Erat Demonstrandum

$$\underline{\mathbf{d}}_3 = \frac{\underline{\mathbf{v}}_3}{|\underline{\mathbf{v}}_3|} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^T$$
$$\underline{\mathbf{d}}_4 = \frac{\underline{\mathbf{v}}_4}{|\underline{\mathbf{v}}_4|} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)^T$$

É claro que $\underline{\mathbf{d}}_i$'s são ortonormais, pois: $\underline{\mathbf{d}}_i \cdot \underline{\mathbf{d}}_j = 0$ por $i \neq j$, e: $\underline{\mathbf{d}}_i \cdot \underline{\mathbf{d}}_i = 1$.

(c) Organizamos os $\underline{\mathbf{d}}_i$'s como colunas em uma matriz:

$$\underline{\underline{\mathbf{D}}} = \begin{pmatrix} | & | & | & | \\ \underline{\mathbf{d}}_1 & \underline{\mathbf{d}}_2 & \underline{\mathbf{d}}_3 & \underline{\mathbf{d}}_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Temos:

$$\underline{\mathbf{x}}_A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\underline{\mathbf{x}}_N$$

(d) Por $\underline{\underline{\mathbf{D}}}$ ser *ortogonal*, temos: $\underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T$. Observamos que $\underline{\underline{\mathbf{D}}}$ ainda é simétrica: $\underline{\underline{\mathbf{D}}} = \underline{\underline{\mathbf{D}}}^T$, ou seja: $\underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}$. Assim, invertemos a equação acima:

$$\underline{\mathbf{x}}_N = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\underline{\mathbf{x}}_A$$

(e) Já justificamos isto no item anterior; por $\underline{\underline{\mathbf{D}}}$ ser ortogonal e simétrica, vale: $\underline{\underline{\mathbf{D}}}' = \underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T = \underline{\underline{\mathbf{D}}}$.

(f) Encontramos $\underline{\mathbf{w}}_1$ em coordenados novos:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1+1-1+1 \\ -1-1-1-1 \\ -1+1+1-1 \\ -1-1+1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

Ilustrando que: $\underline{\mathbf{w}}_1 = -2\underline{\mathbf{d}}_2$.

Para $\underline{\mathbf{w}}_2$:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+2+3+4 \\ 1-2+3-4 \\ 1+2-3-4 \\ 1-2-3+4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -2 \\ 0 \end{pmatrix}$$

Podemos verificar: $\underline{\mathbf{w}}_2 = 5\underline{\mathbf{d}}_1 - \underline{\mathbf{d}}_2 - 2\underline{\mathbf{d}}_3$.

3. *2 pts.* (Cabeludo?) *Ortogonalização de Graham-Schmidt*
Dado os vetores em $\mathbb{R}^3$:

$$\underline{\mathbf{v}}_1 = (1, 1, 1)^T \qquad \underline{\mathbf{v}}_2 = (1, -1, 1)^T \qquad \underline{\mathbf{v}}_3 = (1, 1, -1)^T$$

(a) Mostre que $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$ *não* são mutualmente ortogonais.

(b) Mostre que $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$ são linearmente independentes.

(c) Escolhendo: $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1$ e $\underline{\mathbf{d}}_2 = \underline{\mathbf{v}}_2 + \alpha\underline{\mathbf{d}}_1$, mostre que por:

$$\alpha = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_2}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1}$$

obtemos um vetor, $\underline{\mathbf{d}}_2 \perp \underline{\mathbf{d}}_1$. Encontrar $\underline{\mathbf{d}}_2$.

(d) Escolhendo: $\underline{\mathbf{d}}_3 = \underline{\mathbf{v}}_3 + \beta\underline{\mathbf{d}}_1 + \gamma\underline{\mathbf{d}}_2$, mostre que por:

$$\beta = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1} \qquad \gamma = -\frac{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{d}}_2}$$

obtemos um vetor, $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_2$. Encontrar $\underline{\mathbf{d}}_3$.

Solution:

2 pts. (Cabeludo?) *Ortogonalização de Graham-Schmidt*

Dado os vetores em $\mathbb{R}^3$:

$$\underline{\mathbf{v}}_1 = (1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,-1,1)^T \qquad \underline{\mathbf{v}}_3 = (1,1,-1)^T$$

(a) Mostre que $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$ *não* são mutualmente ortogonais.

(b) Mostre que $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$ são linearmente independentes.

(c) Escolhendo: $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1$ e $\underline{\mathbf{d}}_2 = \underline{\mathbf{v}}_2 + \alpha\underline{\mathbf{d}}_1$, mostre que por:

$$\alpha = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_2}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1}$$

obtemos um vetor, $\underline{\mathbf{d}}_2 \perp \underline{\mathbf{d}}_1$. Encontrar $\underline{\mathbf{d}}_2$.

(d) Escolhendo: $\underline{\mathbf{d}}_3 = \underline{\mathbf{v}}_3 + \beta\underline{\mathbf{d}}_1 + \gamma\underline{\mathbf{d}}_2$, mostre que por:

$$\beta = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1} \qquad \gamma = -\frac{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{d}}_2}$$

obtemos um vetor, $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_2$. Encontrar $\underline{\mathbf{d}}_3$.

Solução:

(a) Os $\underline{\mathbf{v}}_i$'s não são mutualmente ortogonais, pois:

$$\underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 = 1 - 1 + 1 = 1 \neq 0 \quad \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_3 = 1 + 1 - 1 = 1 \neq 0$$

$$\underline{\mathbf{v}}_2 \cdot \underline{\mathbf{v}}_3 = 1 - 1 - 1 = -1 \neq 0$$

(b)

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} | & | & | \\ \underline{\mathbf{v}}_1 & \underline{\mathbf{v}}_2 & \underline{\mathbf{v}}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Segue, que $\rho_{\underline{\underline{\mathbf{V}}}} = 3$, ou seja: os $\underline{\mathbf{v}}_i$'s são linearmente independentes.

(c) Na equação $\underline{\mathbf{d}}_2 = \underline{\mathbf{v}}_2 + \alpha\underline{\mathbf{d}}_1$, fazemos o produto escalar com $\underline{\mathbf{d}}_1$:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_2 = \underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_2 + \alpha(\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1)$$

Exigindo $\underline{\mathbf{d}}_1 \perp \underline{\mathbf{d}}_2$ - ou seja: $\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_2 = 0$ - segue:

$$\alpha = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_2}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1}$$

Usando resultados anteriores:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_2 = 1$$
$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1 = 3$$

Assim: $\alpha = -\frac{1}{3}$, e assim:

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$$\underline{\mathbf{d}}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix}$$

Observamos:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$$

Isto é: $\underline{\mathbf{d}}_2 \perp \underline{\mathbf{d}}_1$.

(d) Na $\underline{\mathbf{d}}_3 = \underline{\mathbf{v}}_3 + \beta\underline{\mathbf{d}}_1 + \gamma\underline{\mathbf{d}}_2$ fazemos o produto escalar com $\underline{\mathbf{d}}_1$ e utilizamos que $\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_2 = 0$:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_3 = \underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_3 + \beta(\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1) + 0$$

Exigindo $\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_3 = 0$, obtemos:

$$\beta = -\frac{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1}$$

Fazendo o produto escalar com $\underline{\mathbf{d}}_2$, segue de mesmo maneira:

$$\gamma = -\frac{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{v}}_3}{\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{d}}_2}$$

Usando resultados anteriores:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{v}}_3 = 1$$
$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_1 = 3$$

Assim:

$$\beta = -\frac{1}{3}$$

E mais:

$$\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{v}}_3 = -\frac{4}{3}$$
$$\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{d}}_2 = \frac{24}{9} = \frac{8}{3}$$

Assim:

$$\gamma = \frac{1}{2}$$

Finalmente calculamos $\underline{\mathbf{d}}_3$:

$$\underline{\mathbf{d}}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Observamos:

$$\underline{\mathbf{d}}_1 \cdot \underline{\mathbf{d}}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$
$$\underline{\mathbf{d}}_2 \cdot \underline{\mathbf{d}}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3} + 0 - \frac{2}{3} = 0$$

Isto é: $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_3 \perp \underline{\mathbf{d}}_2$.

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Data:	17/06/2009
Semestre:	
Curso:	Engenharia Civil
Disciplina:	Álgebra Linear
Prova:	II

1. *2 pts.* Dado os vetores em relação ao base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4, \underline{\mathbf{e}}_5)$, em $\mathbb{R}^5$:

$$\underline{\mathbf{v}}_1 = (1,2,2,2,1)^T \qquad \underline{\mathbf{v}}_2 = (1,0,0,0,1)^T \qquad \underline{\mathbf{v}}_3 = (2,2,2,2,2)^T \qquad \underline{\mathbf{v}}_4 = (0,1,1,1,0)^T$$

(a) Mostre que: $V = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4)$. Qual a dimensão do V?

(b) Escreve $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$ como combinações lineares de $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$.

(c) Escreve $\underline{\mathbf{v}}_3$ e $\underline{\mathbf{v}}_4$ como combinações lineares de $\underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_2$.

Solution:

(a) Formamos o matriz com os vetores em colunas:

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Dai segue: $dim\, ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = dim\, ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = dim\, ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = 2$, o que implica: $ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2) = ger(\underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4) = 2$. A dimensão é o posto do matriz $\underline{\underline{\mathbf{V}}}$, isto é 2.

(b) Omitindo as linhas com somente com zeros e trocando a ordem dos vetores:

$$\left(\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \end{array}\right)$$

Segue: $\underline{\mathbf{v}}_1 = \frac{1}{2}\underline{\mathbf{v}}_3 + \underline{\mathbf{v}}_4$, e: $\underline{\mathbf{v}}_2 = \frac{1}{2}\underline{\mathbf{v}}_3 - \underline{\mathbf{v}}_4$.

(c) Similarmente:

$$\left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$$

Segue: $\underline{\mathbf{v}}_3 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$, e: $\underline{\mathbf{v}}_4 = \frac{1}{2}\underline{\mathbf{v}}_1 - \frac{1}{2}\underline{\mathbf{v}}_2$.

2. *3 pts.* Qual a dimensão dos conjuntos gerados de:

(a) $f_1(x) = x(1-x)$, $f_2(x) = x(1+x)$, $f_3(x) = x(1-x^2)$ e $f_4(x) = x(3-x^2)$.

(b) $\underline{\mathbf{v}}_1 = (1,-1,-1,-1)^T$, $\underline{\mathbf{v}}_2 = (1,1,-1,-1)^T$, $\underline{\mathbf{v}}_3 = (1,1,1,-1)^T$, $\underline{\mathbf{v}}_4 = (1,1,1,1)^T$, $\underline{\mathbf{v}}_5 = (0,0,1,1)^T$.

(c) $\underline{\underline{\mathbf{A}}}_1 = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, $\underline{\underline{\mathbf{A}}}_2 = \begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix}$, $\underline{\underline{\mathbf{A}}}_3 = \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$, $\underline{\underline{\mathbf{A}}}_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution:

(a) Desenvolvendo:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) = c_1 x(1-x) + c_2 x(1+x) + c_3 x(1-x^2) + c_4 x(3-x^2) =$$

$$x\,(c_1 + c_2 + c_3 + 3c_4) + x^2\,(-c_1 + c_2) + x^3\,(c_3 - c_4) \equiv 0$$

O que vale se e somente se: $c_1 + c_2 + c_3 + 3c_4 = -c_1 + c_2 - c_3 - 3c_4 = c_3 - c_4 = 0$. Na forma matricial:

$$\underline{\underline{\mathbf{F}}} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

O que mostre que $\rho_{\underline{\underline{\mathbf{F}}}} = dim\, ger(f_1(x), f_2(x), f_3(x), f_4(x)) = 3$.

(b) Siempre el mismo estribilho:

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

O que mostre que $\rho_{\underline{\underline{\mathbf{V}}}} = dim\ ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4, \underline{\mathbf{v}}_5) = 4$.

(c) Transformamos a equação matricial:

$$c_1 \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix} + c_3 \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} + c_4 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\mathbf{0}}}$$

em uma equação vetorial:

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 3 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} 5 \\ 3 \\ 1 \\ 2 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$$

Com matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & -2 & 5 & 1 \\ 3 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 2 & -4 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & 1 \\ 0 & 7 & -12 & -2 \\ 0 & 7 & -9 & -1 \\ 0 & 0 & -8 & -2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & -2 & 5 & 1 \\ 0 & 7 & -12 & -2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & -8 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & 1 \\ 0 & 7 & -12 & -2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & 1 \\ 0 & 7 & -12 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

O que mostre que $\rho_{\underline{\underline{\mathbf{A}}}} = dim\ ger(\underline{\underline{\mathbf{A}}}_1, \underline{\underline{\mathbf{A}}}_2, \underline{\underline{\mathbf{A}}}_3, \underline{\underline{\mathbf{A}}}_4) = 4$.

3. *3 pts.* Dado os vetores em $\mathbb{R}^3$:

$$\underline{\mathbf{v}}_1 = (1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,1,0)^T \qquad \underline{\mathbf{v}}_3 = (1,0,1)^T$$

(a) Mostre que $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$ formam uma base em $\mathbb{R}^3$.

(b) Encontrar uma relação matricial expressando os coordenados em relação ao base $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$, em termos dos coordenados em relação ao base canônica, $(\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}})$.

(c) Encontrar os coordenados em relação ao base $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$ dos vetores: $\underline{\mathbf{w}}_1 = (1,2,3)^T$ e $\underline{\mathbf{w}}_2 = (3,2,1)^T$

Solution:

3 pts. Dado os vetores em $\mathbb{R}^3$:

$$\underline{\mathbf{v}}_1 = (1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,1,0)^T \qquad \underline{\mathbf{v}}_3 = (1,0,1)^T$$

(a) Mostre que $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$ formam uma base em $\mathbb{R}^3$.

(b) Encontrar uma relação matricial expressando os coordenados em relação ao base $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$, em termos dos coordenados em relação ao base canônica, $(\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}})$.

(c) Encontrar os coordenados em relação ao base $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$ dos vetores: $\underline{\mathbf{w}}_1 = (1,2,3)^T$ e $\underline{\mathbf{w}}_2 = (3,2,1)^T$

Solução:

(a) Formamos o matriz: $\underline{\underline{\mathbf{V}}} = (\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$:

$$\left(\underline{\underline{\mathbf{V}}} \,\middle|\, \underline{\underline{\mathbf{I}}}\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = \left(\underline{\underline{\mathbf{I}}} \,\middle|\, \underline{\underline{\mathbf{V}}}^{-1}\right)$$

Mostrando que $\underline{\underline{\mathbf{V}}}$ é regular e assim $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$ formam um base em $\mathbb{R}^3$.

(b) No cálculo na questão anterior calculamos o inverso, $\underline{\underline{\mathbf{V}}}^{-1}$: $\underline{\mathbf{x}}_A = \underline{\underline{\mathbf{V}}}\underline{\mathbf{x}}_N$, ou equivalentemente: $\underline{\mathbf{x}}_N = \underline{\underline{\mathbf{V}}}^{-1}\underline{\mathbf{x}}_A$:

$$\underline{\mathbf{x}}_N = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \underline{\mathbf{x}}_A$$

(c) Usando esta equação enocntramos os coordenados novos dos vetores $\underline{\mathbf{w}}_1$:

$$\underline{\mathbf{w}}_1' = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$$

E do $\underline{\mathbf{w}}_2$:

$$\underline{\mathbf{w}}_2' = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

4. *2 pts.* Considere os vetores do exercísio anterior:

$$\underline{\mathbf{v}}_1 = (1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,1,0)^T \qquad \underline{\mathbf{v}}_3 = (1,0,1)^T$$

(a) Definindo: $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1$ e $\underline{\mathbf{d}}_2 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{v}}_2$. Encontrar $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$. Mostrar que $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$ são ortogonais.

(b) Definindo: $\underline{\mathbf{d}}_3 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{d}}_2$. Encontrar $\underline{\mathbf{d}}_3$. Mostrar que $\underline{\mathbf{d}}_3$ é ortogonal em $\underline{\mathbf{d}}_1$e $\underline{\mathbf{d}}_2$.

(c) Encontrar uma base ortonormal de $\mathbb{R}^3$, cuja os eixos são paralelos com os vetores $\underline{\mathbf{d}}_1$, $\underline{\mathbf{d}}_2$ e $\underline{\mathbf{d}}_3$. Encontrar o matriz deste substituição ortonormal, $\underline{\underline{\mathbf{M}}}$. Demostrar que $\underline{\underline{\mathbf{M}}}^{-1} = \underline{\underline{\mathbf{M}}}^T$ e encontrar seu determinante.

(d) Encontrar neste base os coordenados dos vetores: $\underline{\mathbf{w}}_1 = (0,1,1)^T$ e $\underline{\mathbf{w}}_2(0,0,1)^T$.

Solution:

2 pts. Considere os vetores do exercísio anterior:

$$\underline{\mathbf{v}}_1 = (1,1,1)^T \qquad \underline{\mathbf{v}}_2 = (1,1,0)^T \qquad \underline{\mathbf{v}}_3 = (1,0,1)^T$$

(a) Definindo: $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1$ e $\underline{\mathbf{d}}_2 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{v}}_2$. Encontrar $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$. Mostrar que $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$ são ortogonais.

(b) Definindo: $\underline{\mathbf{d}}_3 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{d}}_2$. Encontrar $\underline{\mathbf{d}}_3$. Mostrar que $\underline{\mathbf{d}}_3$ é ortogonal em $\underline{\mathbf{d}}_1$e $\underline{\mathbf{d}}_2$.

(c) Encontrar uma base ortonormal de $\mathbb{R}^3$, cuja os eixos são paralelos com os vetores $\underline{\mathbf{d}}_1$, $\underline{\mathbf{d}}_2$ e $\underline{\mathbf{d}}_3$. Encontrar o matriz deste substituição ortonormal, $\underline{\underline{\mathbf{M}}}$. Demostrar que $\underline{\underline{\mathbf{M}}}^{-1} = \underline{\underline{\mathbf{M}}}^T$ e encontrar seu determinante.

(d) Encontrar neste base os coordenados dos vetores: $\underline{\mathbf{w}}_1 = (0,1,1)^T$ e $\underline{\mathbf{w}}_2(0,0,1)^T$.

Solução:

(a) $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1 = (1,1,1)^T$, e:

$$\underline{\mathbf{d}}_1 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{v}}_2 = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (-1,1,0)^T$$

Por ser o produto vetorial (que produz um vetor ortogonal das operantes), $\underline{\mathbf{d}}_1 \perp \underline{\mathbf{d}}_2$.

(b) Calculando:

$$\underline{\mathbf{d}}_3 = \underline{\mathbf{d}}_1 \times \underline{\mathbf{d}}_2 = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1,-1,2)^T$$

De novo, pelos propriedades do produto vetorial, $\underline{\mathbf{d}}_3$ é ortogonal em $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$.

(c) Como os vetores $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3$ são mutualmente ortogonais, basta nos normalizar as três para obter um base ortonormal:

$$\begin{aligned} \underline{\mathbf{m}}_1 &= \frac{\underline{\mathbf{d}}_1}{|\underline{\mathbf{d}}_1|} &=& \tfrac{1}{\sqrt{3}}(1,1,1)^T \\ \underline{\mathbf{m}}_2 &= \frac{\underline{\mathbf{d}}_2}{|\underline{\mathbf{d}}_2|} &=& \frac{1}{\sqrt{2}}(-1,1,0)^T \\ \underline{\mathbf{m}}_3 &= \frac{\underline{\mathbf{d}}_3}{|\underline{\mathbf{d}}_3|} &+=& \frac{1}{\sqrt{6}}(-1,-1,2)^T \end{aligned}$$

Organizando como colunas em uma matriz ortogonal:

$$\underline{\underline{\mathbf{M}}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Por ser ortogonal, temos $\underline{\underline{\mathbf{M}}}^{-1} = \underline{\underline{\mathbf{M}}}^T$ (alternativamente, basta so mostrar que $\underline{\underline{\mathbf{M}}}^T\ \underline{\underline{\mathbf{M}}} = \underline{\underline{\mathbf{I}}}$). E para seu determinante:

$$\det \underline{\underline{\mathbf{M}}} = \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$\frac{1}{6}\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = \frac{1}{6}(6-0)) = 1$$

(d) O matriz tranformando coordenados antigos em coordenados novos, é $\underline{\underline{\mathbf{M}}}^{-1}$, assim:

$$\underline{\mathbf{w}}_1' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

E:

$$\underline{\mathbf{w}}_2' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

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Curso:	Engenharia Civil
Disciplina:	Álgebra Linear
Prova:	III - Chamada Extra

1. (465) *4 pts.* Dado o matriz, $\underline{\underline{\mathbf{A}}}$, de uma aplicação linear, $f : \mathbb{R}^4 \to \mathbb{R}^4$:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 2 & 3 & 0 & 3 \\ -2 & -1 & 2 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

(a) Mostre que o núcleo (kernel), $ker f = \{\underline{\mathbf{x}} \in \mathbb{R}^4 | \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$, tem dimensão 0.

(b) Encontrar autovalores e autovetores de f.

(c) Mostre que é possível escolher um base de autovetores de f.

(d) Encontre o matriz, $\underline{\underline{\mathbf{B}}}$, de f ao respeito desde base. Qual a relação entre $\underline{\underline{\mathbf{A}}}$ e $\underline{\underline{\mathbf{B}}}$?

Solution:

(a)

$$\det \underline{\underline{\mathbf{A}}} = \begin{vmatrix} 1 & 0 & 0 & -3 \\ 2 & 3 & 0 & 3 \\ -2 & -1 & 2 & -3 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 4 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -1 & 2 \end{vmatrix} = 4 \cdot 1 \cdot 3 \cdot 2 = 24 \neq 0$$

Ou seja, $\underline{\underline{\mathbf{A}}}$ é regular, isto é: $dim\ ker\ f = 0$.

(b)

$$P(\lambda) = \det(\underline{\underline{\mathbf{A}}} - \lambda \underline{\underline{\mathbf{I}}}) = \begin{vmatrix} 1-\lambda & 0 & 0 & -3 \\ 2 & 3-\lambda & 0 & 3 \\ -2 & -1 & 2-\lambda & -3 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ -2 & -1 & 2-\lambda \end{vmatrix} =$$

$$(4-\lambda)(1-\lambda)(3-\lambda)(2-\lambda) = 0 \quad \Leftrightarrow \quad \lambda = 1 \vee \lambda = 2 \vee \lambda = 3 \vee \lambda = 4$$

Notamos que a soma dos autovalores é igual o traço do $\underline{\underline{\mathbf{A}}}$.

i. $\lambda_1 = 1$:

$$\underline{\underline{\mathbf{A}}} - \underline{\underline{\mathbf{I}}} = \begin{pmatrix} 0 & 0 & 0 & -3 \\ 2 & 2 & 0 & 3 \\ -2 & -1 & 1 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_1 = x_3 = -x_2 \wedge x_4 = 0 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(1, -1, 1, 0)^T, t \in \mathbb{R}$$

ii. $\lambda_2 = 2$:

$$\underline{\underline{\mathbf{A}}} - 2\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -1 & 0 & 0 & -3 \\ 2 & 1 & 0 & 3 \\ -2 & -1 & 0 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_1 = x_2 = x_4 = 0 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(0, 0, 1, 0)^T, t \in \mathbb{R}$$

iii. $\lambda_3 = 3$:

$$\underline{\underline{\mathbf{A}}} - 3\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -2 & 0 & 0 & -3 \\ 2 & 0 & 0 & 3 \\ -2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_2 = -x_3 \wedge x_1 = x_4 = 0 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(0,1,-1,0)^T, t \in \mathbb{R}$$

iv. $\lambda_4 = 4$:

$$\underline{\underline{\mathbf{A}}} - 4\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -3 & 0 & 0 & -3 \\ 2 & -1 & 0 & 3 \\ -2 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 3 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_2 = x_4 \wedge x_1 = -x_4 \wedge x_3 = -x_4 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(-1,1,-1,1)^T, t \in \mathbb{R}$$

(c) Por ter 4 autovalores diferentes, é possível escolher uma base de autovetores, por ex:

$$\underline{\mathbf{v}}_1 = (1,-1,1,0) \quad \underline{\mathbf{v}}_2 = (0,0,1,0) \quad \underline{\mathbf{v}}_3 = (0,1,-1,0) \quad \underline{\mathbf{v}}_4 = (-1,1,-1,1)$$

(d) Ao respeito da base $\underline{\mathbf{v}}_i$ a matriz é:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Colocando os $\underline{\mathbf{v}}_i$'s nos colunas do matriz $\underline{\underline{\mathbf{V}}}$:

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A relação entre $\underline{\underline{\mathbf{A}}}$ e $\underline{\underline{\mathbf{B}}}$ é: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{V}}}^{-1}\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{V}}}$.

Ressaltamos que $\underline{\underline{\mathbf{V}}}^{-1} \neq \underline{\underline{\mathbf{V}}}$, pois $\underline{\underline{\mathbf{V}}}$ não é ortogonal (ie. os $\underline{\mathbf{v}}_i$'s não são ortonormais - nem ortogonais).

2. (341, c) *4 pts.* Dado a forma quadrática:

$$(*) \; x^2 + y^2 - z^2 + 2xy$$

(a) Encontrar uma substituição ortogonal, $\underline{\underline{\mathbf{D}}}$, que reduz $(*)$ em uma forma sem termos mistos: $\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 x z_1^2$ - onde $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

(b) Classificar geométricalmente: $x^2 + y^2 - z^2 + 2xy - 2x - 4y - 1 = 0$.

Solution:

(a) Primeiramente, com $\underline{\mathbf{r}} = (x,y,z)^T$:

$$x^2 + y^2 - z^2 + 2xy = \underline{\mathbf{r}}^T \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{r}} = (x \; y \; z) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Procuramos os autovalores do $\underline{\underline{\mathbf{A}}}$:

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda \underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0 \Leftrightarrow$$

$$\lambda = -1 \vee (1-\lambda)^2 = 1 \quad \Leftrightarrow \quad \lambda = -1 \vee 1-\lambda = \pm 1 \quad \Leftrightarrow \quad \lambda_1 = 2 \vee \lambda_2 = 0 \vee \lambda_3 = -1$$

Notamos: $\lambda_1 + \lambda_2 + \lambda_3 = 1 = Tr\, \underline{\underline{\mathbf{A}}}$. E: $\lambda_1\lambda_2\lambda_3 = 0 = \det \underline{\underline{\mathbf{A}}}$.

Encontramos autovetores:

i. $\lambda_1 = 2$:

$$\underline{\underline{\mathbf{A}}} - 0\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_1 = x_2 \wedge x_3 = 0 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(1,1,0)^T, t \in \mathbb{R}$$

Normalizando: $\underline{\mathbf{v}}_1 = \frac{1}{\sqrt{2}}(1,1,0)^T$.

ii. $\lambda_2 = 0$:

$$\underline{\underline{\mathbf{A}}} - \underline{\underline{\mathbf{I}}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow$$

$$x_1 = -x_2 \wedge x_3 = 0 \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(-1,1,0)^T, t \in \mathbb{R}$$

Normalizando: $\underline{\mathbf{v}}_2 = \frac{1}{\sqrt{2}}(-1,1,0)^T$.

iii. $\lambda_3 = -1$: Por ser ortogonal dos outros dois autovetores, temos:

$$(1,1,0)^T \times (-1,-,0)^T = (0,0,2)^T$$

Normalizando: $\underline{\mathbf{v}}_3 = (0,0,1)^T$.

Assim, a substituição ortogonal diagonalizando $(*)$ é:

$$\underline{\underline{\mathbf{V}}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{\mathbf{V}}}^{-1} = \underline{\underline{\mathbf{V}}}^T$$

A matriz neste base é:

$$\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{V}}}^T \underline{\underline{\mathbf{A}}}\, \underline{\underline{\mathbf{V}}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Transformando $(*)$ em:

$$x^2 + y^2 - z^2 + 2xy = 2x'^2 - z'^2$$

A relação entre os coordenados antigos e novos é:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\underline{\mathbf{V}}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x'-y') \\ \frac{1}{\sqrt{2}}(x'+y') \\ z' \end{pmatrix}$$

(b) Usando o resultado do item anterior:

$$-2x - 4y = -\frac{1}{\sqrt{2}}\left[2(x'-y') + 4(x'+y')\right] = -\frac{1}{\sqrt{2}}(6x' + 2y')$$

Assim:

$$x^2 + y^2 - z^2 + 2xy - 2x - 4y - 1 = 0 \quad \Leftrightarrow$$

$$2x'^2 - z'^2 - \frac{1}{\sqrt{2}}(6x' + 2y') = 1 \quad \Leftrightarrow$$

$$2\left(x'^2 - \frac{3}{\sqrt{2}}x'\right) - z'^2 - \frac{2}{\sqrt{2}}y' = 1 \quad \Leftrightarrow$$

$$2\left((x' - \frac{3}{2\sqrt{2}})^2 - \frac{9}{8}\right) - z'^2 - \sqrt{2}y' = 1 \quad \Leftrightarrow$$

$$2(x' - \frac{3}{2\sqrt{2}})^2 - z'^2 - \sqrt{2}y' = 1 + \frac{9}{4} = \frac{13}{4} \quad \Leftrightarrow$$

$$(x' - \frac{3}{2\sqrt{2}})^2 - \frac{z'}{\sqrt{2}}^2 = \frac{y'}{\sqrt{2}} + \frac{13}{8} = \frac{y' + \frac{13}{4\sqrt{2}}}{\sqrt{2}}$$

Isto é uma paraboloide hiperbólica, com centro:
$(x'_0, y'_0, z'_0) = (\frac{3}{2\sqrt{2}}, -\frac{13}{4\sqrt{2}}, 0)$. Em os coordenados antigos:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2\sqrt{2}} \\ -\frac{13}{4\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + \frac{13}{8} \\ \frac{3}{4} - \frac{13}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{19}{8} \\ -\frac{7}{8} \\ 0 \end{pmatrix}$$

Quem diria...

3. (471) *2 pts.* (Projeção ortogonal.) Uma aplicação linear, f, é dado por:

$$f(\underline{\mathbf{x}}) = (\underline{\mathbf{x}} \cdot \underline{\mathbf{e}})\underline{\mathbf{e}} - \underline{\mathbf{x}}$$

onde $\underline{\mathbf{e}}$ é um vetor de unidade dado (fixo).

(a) Fazer uma figura indicando os vetores $\underline{\mathbf{e}}, \underline{\mathbf{x}}$ e $f(\underline{\mathbf{x}})$.

(b) Mostre que a imagem do f é perpendicular em $\underline{\mathbf{e}}$.

(c) Seja $\underline{\mathbf{i}}$ e $\underline{\mathbf{j}}$ dois vetores unitários e ortogonais. Pondo: $\underline{\mathbf{e}} = \frac{\sqrt{2}}{2}\underline{\mathbf{i}} + \frac{\sqrt{2}}{2}\underline{\mathbf{j}}$. Encontrar o matriz, $\underline{\underline{\mathbf{A}}}$, de f em relação ao base $(\underline{\mathbf{i}}, \underline{\mathbf{j}})$.

Solution:

(a) -

(b) $f(\underline{\mathbf{x}}) \cdot \underline{\mathbf{e}} = ((\underline{\mathbf{x}} \cdot \underline{\mathbf{e}})\underline{\mathbf{e}} - \underline{\mathbf{x}}) \cdot \underline{\mathbf{e}} = (\underline{\mathbf{x}} \cdot \underline{\mathbf{e}})(\underline{\mathbf{e}} \cdot \underline{\mathbf{e}}) - (\underline{\mathbf{x}} \cdot \underline{\mathbf{e}}) = (\underline{\mathbf{x}} \cdot \underline{\mathbf{e}}) - (\underline{\mathbf{x}} \cdot \underline{\mathbf{e}}) = 0.$

(c) $f(\underline{\mathbf{i}}) = (\underline{\mathbf{i}} \cdot \underline{\mathbf{e}})\underline{\mathbf{e}} - \underline{\mathbf{i}} = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}\underline{\mathbf{i}} + \frac{\sqrt{2}}{2}\underline{\mathbf{j}}) - \underline{\mathbf{i}} = \frac{1}{2}\underline{\mathbf{i}} + \frac{1}{2}\underline{\mathbf{j}} - \underline{\mathbf{i}} = (-\frac{1}{2}, \frac{1}{2}),$
$f(\underline{\mathbf{j}}) = (\underline{\mathbf{j}} \cdot \underline{\mathbf{e}})\underline{\mathbf{e}} - \underline{\mathbf{j}} = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2}\underline{\mathbf{i}} + \frac{\sqrt{2}}{2}\underline{\mathbf{j}}) - \underline{\mathbf{j}} = \frac{1}{2}\underline{\mathbf{i}} + \frac{1}{2}\underline{\mathbf{j}} - \underline{\mathbf{j}} = (\frac{1}{2}, -\frac{1}{2}).$
Assim o matriz é:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

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Disciplina: Álgebra Linear
Prova: III

1. (410) *4 pts.* Em $\mathbb{R}^4$ são dado os vetores:

$$\underline{\mathbf{d}}_1 = (1,2,2,0)^T \quad \underline{\mathbf{d}}_2 = (0,1,1,1)^T \quad \underline{\mathbf{d}}_3 = (0,0,1,1)^T \quad \underline{\mathbf{d}}_4 = (1,1,1,1)^T$$

   (a) Mostre que $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4$ formam uma base em $\mathbb{R}^4$.

   (b) Uma aplicação linear, $f : \mathbb{R}^4 \to \mathbb{R}^3$ é dado por:

$$f(\underline{\mathbf{d}}_1) = (1,1,2)^T \quad f(\underline{\mathbf{d}}_2) = (3,-1,1)^T \quad f(\underline{\mathbf{d}}_3) = (4,0,3)^T \quad f(\underline{\mathbf{d}}_4) = (-5,3,0)^T$$

   Encontrar a matriz do f em respeito ao base $\underline{\mathbf{d}}_i$ em $\mathbb{R}^4$ e a base canônica em $\mathbb{R}^3$.

   (c) Encontrar a matriz do f em respeito ao base canônica em $\mathbb{R}^4$ e $\mathbb{R}^3$.

   (d) Encontrar a dimensão do imagem do f.

   (e) Dados os vetores: $\underline{\mathbf{v}}_1 = \underline{\mathbf{d}}_1 + \underline{\mathbf{d}}_2 - \underline{\mathbf{d}}_3$ e $\underline{\mathbf{v}}_2 = -\underline{\mathbf{d}}_1 + 2\underline{\mathbf{d}}_2 + \underline{\mathbf{d}}_4$.
   Mostre que: $ker f = \left\{\underline{\mathbf{x}} \in \mathbb{R}^4 | \, f(\underline{\mathbf{x}}) = \underline{\mathbf{0}}\right\} = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2)$.

   (f) Encontrar a solução completa da equação: $f(\underline{\mathbf{x}}) = f(\underline{\mathbf{d}}_1)$.

Solution:

(410) *4 pts.* Em $\mathbb{R}^4$ são dado os vetores:

$$\underline{\mathbf{d}}_1 = (1,2,2,0)^T \quad \underline{\mathbf{d}}_2 = (0,1,1,1)^T \quad \underline{\mathbf{d}}_3 = (0,0,1,1)^T \quad \underline{\mathbf{d}}_4 = (1,1,1,1)^T$$

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \Rightarrow \rho = 4$$

Ou seja, $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4$ formam uma base em $\mathbb{R}^4$.

(b) Em respeito ao base canônica em $\mathbb{R}^4$ e $\mathbb{R}^3$ o matriz é (a imagem dos $\underline{\mathbf{d}}_i$ em colunas):

$$\underline{\underline{\mathbf{A}}}' = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}$$

(c) Temos: $\underline{\underline{\mathbf{A}}}' = \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{D}}} \quad \Leftrightarrow \quad \underline{\underline{\mathbf{A}}} = \underline{\underline{\mathbf{A}}}'\underline{\underline{\mathbf{D}}}^{-1}$. Calculamos a matriz inversa:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \sim$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad \Rightarrow \quad \underline{\underline{\mathbf{D}}}^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Multiplicando:

$$\underline{\underline{\mathbf{A}}} = \underline{\underline{\mathbf{A}}}'\underline{\underline{\mathbf{D}}}^{-1} = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -8 & -1 & \frac{11}{2} & -\frac{3}{2} \\ 4 & -1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & -2 & \frac{7}{2} & -\frac{1}{2} \end{pmatrix}$$

(d) A dimensão da imagem é o posto: $\rho_A = \rho_{A'}$ (pois mudar de base não altera a dimensão). Escolhemos $\underline{\underline{\mathbf{A}}}'$:

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & -4 & -4 & 8 \\ 0 & -5 & -5 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Segue que a dimensão da imagem é 2.

(e) Primeiramente: $dim\mathbb{R}^4 = dim\ Im(f) + dim\ ker\ f$. Ou seja: $dim\ ker\ f = 4 - 2 = 2$.

Formamos as imagens:

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+3-4 & -1+6-5 \\ 1-1 & -1-2+3 \\ 2+1-3 & -2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Segue, que $f(\underline{\mathbf{v}}_1) = f(\underline{\mathbf{v}}_2) = \underline{\mathbf{0}}$, ou seja: $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in kerf$. Por $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$ serem linearmente independentes, concluimos: $kerf = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$.

(f) Encontrar a solução completa da equação: $f(\underline{\mathbf{x}} = f(\underline{\mathbf{d}}_1)$.

Trivialmente a quação $f(\underline{\mathbf{x}} = f(\underline{\mathbf{d}}_1)$ tem a solução $\underline{\mathbf{x}} = \underline{\mathbf{d}}_1$, assim sendo uma solução particular. Assim a solução completa é $\underline{\mathbf{x}} = \underline{\mathbf{d}}_1 + kerf$, ou seja:

$$\underline{\mathbf{x}} = \underline{\mathbf{d}}_1 + t\underline{\mathbf{v}}_1 + s\underline{\mathbf{v}}_2,\ t, s \in \mathbb{R}$$

2. (403) *4 pts.* Dado a superfície:

$$(*)\ \ 3x^2 - 3y^2 + 12xz + 12yz + 4x - 4y - 2z = 0$$

(a) Encontrar a parte linear do $(*)$, $F_1(x, y, z)$.

(b) Encontrar a parte quadrática do $(*)$, $F_2(x, y, z)$, e escreve-a na forma matricial: $\underline{\mathbf{r}}^T \underline{\underline{\mathbf{A}}}\ \underline{\mathbf{r}}$.

(c) Encontrar autovalores e autovetores da matriz $\underline{\underline{\mathbf{A}}}$.

(d) Encontrar uma base ortonormal, $\underline{\mathbf{d}}_i$, de autovetores da $\underline{\underline{\mathbf{A}}}$.

(e) Encontrar uma substituição ortogonal, $\underline{\underline{\mathbf{D}}}$, e uma matriz diagonal, $\underline{\underline{\mathbf{B}}}$, tal que: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{D}}}^T \underline{\underline{\mathbf{A}}}\ \underline{\underline{\mathbf{D}}}$.

(f) Transformar, usando o item anterior, F_2 em uma forma quadrática sem termos mistos.

(g) Encontrar $F_1(x, y, z)$ em termos dos coordenados novos.

(h) Classificar $(*)$ geometricalmente.

Solution:

(a) $F_1(x, y, z) = 4x - 4y - 2z$.

(b) $F_2(x,y,z) = 3x^2 - 3y^2 + 12xz + 12yz$. De forma matricial:

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & 0 & 6 \\ 0 & -3 & 6 \\ 6 & 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(c)

$$\begin{vmatrix} 3-\lambda & 0 & 6 \\ 0 & -3-\lambda & 6 \\ 6 & 6 & -\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} -3-\lambda & 6 \\ 6 & -\lambda \end{vmatrix} + 6 \begin{vmatrix} 0 & -3-\lambda \\ 6 & 6 \end{vmatrix}$$

$$= (3-\lambda)\,[\lambda(-3-\lambda) - 36] + 6\,[0 - 6(-3-\lambda)] = ... = -\lambda^3 + 81\lambda = -\lambda(\lambda^2 - 81) = 0 \quad \Leftrightarrow$$

$$\lambda_1 = 9 \vee \lambda_2 = -9 \vee \lambda_3 = 0$$

(d) Encontrar uma base ortonormal, $\underline{\mathbf{d}}_i$, de autovetores da $\underline{\underline{\mathbf{A}}}$.

i. $\underline{\lambda_1 = 9}$:

$$\underline{\underline{\mathbf{A}}} - 9\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -6 & 0 & 6 \\ 0 & -12 & 6 \\ 6 & 6 & -9 \end{pmatrix} \sim ... \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow x_1 = x_3 = 2x_2 \Leftrightarrow$$

$$\underline{\mathbf{x}} = t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \; t \in \mathbb{R}$$

Normalizamos: $\underline{\mathbf{v}}_1 = \frac{1}{3}(2,1,2)$.

ii. $\underline{\lambda_2 = -9}$:

$$\underline{\underline{\mathbf{A}}} - 9\underline{\underline{\mathbf{I}}} = \begin{pmatrix} 12 & 0 & 6 \\ 0 & 6 & 6 \\ 6 & 6 & 9 \end{pmatrix} \sim ... \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow x_3 = -x_2 \wedge x_3 = -2x_1 \Leftrightarrow$$

$$\underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \; t \in \mathbb{R}$$

Normalizamos: $\underline{\mathbf{v}}_2 = \frac{1}{3}(1,2,-2)$. Verificamos: $\underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 = 0$.

iii. $\underline{\lambda_2 = 0}$:
Podemos resolver como para os autovalores anteriores, porém é mais facil utilizar que $\underline{\mathbf{v}}_3 \perp \underline{\mathbf{v}}_1$ e $\underline{\mathbf{v}}_3 \perp \underline{\mathbf{v}}_2$, ou seja: $\underline{\mathbf{v}}_3 \parallel \underline{\mathbf{v}}_1 \times \underline{\mathbf{v}}_2$:

$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} = (-6,6,3) \parallel (-2,2,1)$$

Normalizamos: $\underline{\mathbf{v}}_3 = \frac{1}{3}(-2,2,1)$.
Suggestion: Verifique que $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{v}}_i = \lambda_i \underline{\mathbf{v}}_i$.

(e) Organizando os autovetores normalizados em colunas:

$$\underline{\underline{\mathbf{D}}} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix} \qquad \underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix}$$

O matrix neste base é os autovalores na diagonal:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E vale: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{D}}}^T \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{D}}}$.

(f) $F_2(x,y,x) = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 = 9x'^2 - 9y'^2$.

(g) Usamos a relação entre coordenados velhos e coordenados novos: $\underline{\mathbf{r}} = \underline{\underline{\mathbf{D}}}\,\underline{\mathbf{r}}'$:

$$\underline{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2x'+y'-2z' \\ x'+2y'+2z' \\ 2x'-2y'+z' \end{pmatrix}$$

Assim: $F_1(x,y,z) = \frac{1}{3}\left(4(2x'+y'-2z') - 4(x'+2y'+2z') - 2(2x'-2y'+z')\right) =$
$\frac{1}{3}\left((8-4-4)x' + (4-8+4)y' + (-8-8-2)z'\right) = -6z'$ e eixos paralelas com os vetores $\underline{\mathbf{v}}_1$, $\underline{\mathbf{v}}_2$ e $\underline{\mathbf{v}}_3$.

(h) $(*)$ é equivalente à:

$$9x'^2 - 9y'^2 - 6z' = 0 \quad \Leftrightarrow \quad 6z' = 9x'^2 - 9y'^2$$

Isto é uma paraboloide hiperbólica com centro em $(x',y',z') = (x,y,z) = (0,0,0)$.

3. (442) *2 pts.* Seja $\underline{\mathbf{a}}$ e $\underline{\mathbf{b}}$ vetores fixos em $\mathbb{R}^3$ que satisfaz:

$$|\underline{\mathbf{a}}| = |\underline{\mathbf{b}}| = \sqrt{2} \qquad \underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = 1$$

A aplicação, f, é dado por:

$$f(\underline{\mathbf{x}}) = \underline{\mathbf{a}} \times \underline{\mathbf{x}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{x}})\underline{\mathbf{b}}$$

(a) Mostrar que f é uma aplicação linear.
No resto deste exercísio, pomos: $\underline{\mathbf{c}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}$

(b) Mostre $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}$ formam uma base em $\mathbb{R}^3$. Mostre que o matrix ao respeito desde base é dado por:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

Informamos, que para o produto vetorial duplo, vale:

$$\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}})\underline{\mathbf{b}} - (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{c}}$$

(c) Encontrar autovalores e autovetores do f.

(d) Encontrar a dimensão da imagem e uma base da mesma.

Solution:

(a)
i. $f(\underline{\mathbf{x}}+\underline{\mathbf{y}}) = \underline{\mathbf{a}} \times (\underline{\mathbf{x}}+\underline{\mathbf{y}}) + (\underline{\mathbf{a}} \cdot (\underline{\mathbf{x}}+\underline{\mathbf{y}}))\underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{x}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{x}})\underline{\mathbf{b}} + \underline{\mathbf{a}} \times \underline{\mathbf{y}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{y}})\underline{\mathbf{b}} = f(\underline{\mathbf{x}}) + f(\underline{\mathbf{y}})$,
ii. $f(\alpha\underline{\mathbf{x}}) = \underline{\mathbf{a}} \times (\alpha\underline{\mathbf{x}}) + (\underline{\mathbf{a}} \cdot (\alpha\underline{\mathbf{x}}))\underline{\mathbf{b}} = \alpha\left[\underline{\mathbf{a}} \times \underline{\mathbf{x}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{x}})\underline{\mathbf{b}}\right] = \alpha f(\underline{\mathbf{x}})$,

O que mostre que f é linear.

(b) Sendo $\underline{\mathbf{a}}$ e $\underline{\mathbf{b}}$ linearmente independentes, o vetor $\underline{\mathbf{c}}$ é ortogonal dos dois, e assim $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}$ são linearmente independentes.

Para encontrar o matriz desde base, procuramos as imagens dos vetores básicas:

i. $f(\underline{\mathbf{a}}) = \underline{\mathbf{a}} \times \underline{\mathbf{a}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{a}})\underline{\mathbf{b}} = (\underline{\mathbf{a}} \cdot \underline{\mathbf{a}})\underline{\mathbf{b}} = 2\underline{\mathbf{b}} = [0,2,0]$
ii. $f(\underline{\mathbf{b}}) = \underline{\mathbf{a}} \times \underline{\mathbf{b}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{b}} = \underline{\mathbf{c}} + \underline{\mathbf{b}} = [0,1,1]$
iii. $f(\underline{\mathbf{c}}) = \underline{\mathbf{a}} \times \underline{\mathbf{c}} + (\underline{\mathbf{a}} \cdot \underline{\mathbf{c}})\underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{c}} = \underline{\mathbf{a}} \times (\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{a}} - (\underline{\mathbf{a}} \cdot \underline{\mathbf{a}})\underline{\mathbf{b}} = \underline{\mathbf{a}} - 2\underline{\mathbf{b}} = [1,-2,0]$

Pondo estes resultados em colunas, obtemos o matriz do f:

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

(c)

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 2 & 1-\lambda & -2 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1-\lambda \\ 0 & 1 \end{vmatrix} = -\lambda(\lambda^2-\lambda+2)+2 =$$

$$-\lambda^3+\lambda^2-2\lambda+2 = -\lambda^2(\lambda-1)-2(\lambda-1) = (-\lambda^2-2)(\lambda-1) = 0$$

Assim, temos somente uma autovalor real, sendo: $\lambda = 1$. Calculando os autovetores de $\lambda = 1$:

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \quad \Leftrightarrow \quad \underline{\mathbf{x}} = t(1,1,1)^T,\ t \in \mathbb{R}$$

(d) Por $\lambda = 0$ *não* ser autovalor, temos que o núcleo do f contém somente a vetor nula, ou seja: $dim\ ker\ f = 0$. Assim: $dim\ Im\ f = 3 - 0 = 3$.

Como uma base da imagem podiamos usar $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, pois eles são linearmente independentes, cf. item (a).

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Curso:	*
Disciplina:	Álgebra Linear
Prova:	I

1. (390) *4 pts.* Dado o matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

   (a) Encontrar $\det \underline{\underline{\mathbf{A}}}$ para qualquer valor de λ.

   (b) Para quais valores de λ $\underline{\underline{\mathbf{A}}}$ é singular?

   (c) Para $\lambda = 1$ encontrar o matriz adjunto de $\underline{\underline{\mathbf{A}}}$.

   (d) Para $\lambda = 2$ encontrar o matriz inversa de $\underline{\underline{\mathbf{A}}}$.

   (e) Para $\lambda = 0$ resolver o sistema homogêneo $(1) : \underline{\underline{\mathbf{A}}}\ \underline{\mathbf{x}} = \underline{\mathbf{0}}$. Encontrar a dimensão e uma base desde espaço solucional.

   (f) Para $\lambda = 3$ resolver o sistema homogêneo $(2) : \underline{\underline{\mathbf{A}}}\underline{\mathbf{x}} = \underline{\mathbf{0}}$. Encontrar a dimensão e uma base desde espaço solucional.

   (g) Mostre que qualquer vetor do espaço solucional de (1) é ortogonal em qualquer vetor do espaço solucional de (2).

Solution:

(a)

$$P(\lambda) = \det \underline{\underline{\mathbf{A}}} = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -1 & -1 \\ \lambda-3 & 3-\lambda & 0 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 2-\lambda \end{vmatrix} =$$

$$(3-\lambda) \begin{vmatrix} 2-\lambda & 1-\lambda & -1 \\ -1 & 0 & -0 \\ -1 & -2 & 2-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = (3-\lambda)\left[(1-\lambda)(2-\lambda) - 2\right] = -\lambda(3-\lambda)^2$$

(b)

$$\det \underline{\underline{\mathbf{A}}} = -\lambda(3-\lambda)^2 = 0 \Leftrightarrow \lambda = 0 \vee \lambda = 3$$

(c) Calculando os subdeterminantes:

$$(D_{ij}) = \begin{pmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

E os co-fatores:

$$(A_{ij}) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Este matriz é o adjunto, pois por $\underline{\underline{\mathbf{A}}}$ ser simétrico, não precisamos transpor.

(d)

$$\left(\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{I}}}\right) = \left(\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) = \left(\underline{\underline{\mathbf{I}}}|\underline{\underline{\mathbf{A}}}^{-1}\right)$$

(e)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Ou seja: a dimensão do espaço solucional é 1 e uma base da mesma: $\underline{\mathbf{v}}_1 = (1,1,1)^T$.

(f)

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\mathbf{x}} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

Ou seja: a dimensão do espaço solucional é 2 e uma base da mesma: $\underline{\mathbf{v}}_2 = (-1,0,1)^T$ e $\underline{\mathbf{v}}_3 = (-1,1,0)^T$.

(g) E somente notar, que: $\underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_2 = \underline{\mathbf{v}}_1 \cdot \underline{\mathbf{v}}_3 = 0$.

2. (387) *2 pts.* Dado os vetores:

$$\underline{\mathbf{a}}_1 = (0,1,2,2,0)^T \quad \underline{\mathbf{a}}_2 = (1,1,4,0,0)^T \quad \underline{\mathbf{a}}_3 = (1,2,6,2,1)^T \quad \underline{\mathbf{a}}_4 = (-1,2,2,6,-1)^T$$

(a) Mostrar que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$ são linearmente independentes.

(b) Escrever $\underline{\mathbf{a}}_4$ como uma combinação linear de $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$

Solution:

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 2 \\ 2 & 4 & 6 & 2 \\ 2 & 0 & 2 & 6 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Na redução anterior as primeiras três colunas mostram que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$ são linearmente independentes.

(b) Da última coluna segue: $\underline{\mathbf{a}}_4 = 4\underline{\mathbf{a}}_1 - \underline{\mathbf{a}}_3$

3. (371) *4 pts.* Dado o matriz e o vetor::

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1+a & 0 & 1 \end{pmatrix}, \quad a \in \mathbb{R} \qquad \begin{pmatrix} 0 \\ b \\ 0 \\ b \end{pmatrix}, \quad b \in \mathbb{R}$$

Considerando o sistema linear:

$$(*) \quad \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

(a) Encontrar $\det \underline{\underline{\mathbf{A}}}$ para qualquer $a \in \mathbb{R}$.

(b) Encontrar o posto do matriz $\underline{\underline{\mathbf{A}}}$ para qualquer $a \in \mathbb{R}$.

(c) Encontrar o posto do matriz total do sistema $(*)$ para qualquer $a, b \in \mathbb{R}$ e no cada caso a dimensão do espaço solucional.

(d) Resolver o sistema $(*)$ para quaisquer $a, b \in \mathbb{R}$.

Solution:

(a)

$$\det \underline{\underline{\mathbf{A}}} = \begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1+a & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1-a & 0 \\ 0 & 0 & 0 & -2a-1 \end{vmatrix} = (a-1)(2a+1)$$

(b) Da reduzão anterior segue:

$$\rho_{\underline{\underline{A}}} = \begin{cases} 3 & a = 1 \vee a = -\frac{1}{2} \\ 4 & a \neq 1 \wedge\ a \neq -\frac{1}{2} \end{cases}$$

(c)

$$\underline{\underline{T}} = \begin{pmatrix} 1 & 0 & -a & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ -1 & 0 & 1 & 0 & | & 0 \\ 0 & 1+a & 0 & 1 & | & b \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -a & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ 0 & 0 & 1-a & 0 & | & 0 \\ 0 & 0 & 0 & 2a+1 & | & ab \end{pmatrix}$$

$$\rho_{\underline{\underline{T}}} = \begin{cases} 3 & a = 1 \wedge b \in \mathbb{R} \\ 3 & a = -\frac{1}{2} \wedge b = 0 \\ 4 & a = -\frac{1}{2} \wedge b \neq 0 \\ 4 & a \neq 1 \wedge\ a \neq -\frac{1}{2} \wedge b \in \mathbb{R} \end{cases}$$

(d) Caso I: $a = 1 \wedge b \in \mathbb{R}$.
$n - \rho = 4 - 3 = 1$ parâmetros.

$$\underline{\underline{T}} = \begin{pmatrix} 1 & 0 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 3 & | & b \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ 0 & 0 & 0 & 1 & | & \frac{b}{3} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & \frac{b}{3} \\ 0 & 0 & 0 & 1 & | & \frac{b}{3} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Pondo: $x_3 = t$ obtemos: $x_1 = t$ e $x_2 = x_4 = \frac{b}{3}$, ou seja:

$$\underline{x} = (0, \frac{b}{3}, 0, \frac{b}{3})^T + t(1, 0, 1, 0)^T, \quad t \in \mathbb{R}$$

Caso II: $a = -\frac{1}{2} \wedge b = 0$.
$n - \rho = 4 - 3 = 1$ parâmetros.

$$\underline{\underline{T}} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & 0 \\ 0 & 0 & \frac{3}{2} & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Pondo $x_4 = t$ obtemos: $x_2 = -2t$ e $x_1 = x_3 = 0$, ou seja:

$$\underline{x} = t(0, -2, 0, 1)^T, \quad t \in \mathbb{R}$$

Caso III: $a = -\frac{1}{2} \wedge b \neq 0$.
Neste caso não ha solução, pois: $\rho_{\underline{\underline{A}}} = 3 < \rho_{\underline{\underline{T}}} = 4$.
Caso IV: $a \neq \wedge\, a \neq -\frac{1}{2} \wedge b \in \mathbb{R}$.
$n - \rho = 4 - 4 = 0$ parâmetros, isto é: solução única:

$$\begin{pmatrix} 1 & 0 & -a & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ 0 & 0 & 1-a & 0 & | & 0 \\ 0 & 0 & 0 & 2a+1 & | & ab \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -a & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & b \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & \frac{ab}{2a+1} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & \frac{b}{2a+1} \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & \frac{ab}{2a+1} \end{pmatrix}$$

Assim:

$$\underline{x} = \left(0, \frac{b}{2a+1}, 0, \frac{ab}{2a+1}\right)^T = \frac{b}{2a+1}(0, 1, 0, a)^T$$

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Disciplina:	Álgebra Linear
Prova:	I - 2^a Chamada

1. *4 pts.* Dado o matriz, $\underline{\underline{A}}$, e o vetor, $\underline{b}$:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & a-1 & 2 & a+2 \\ 1 & 2a & 0 & a \\ 0 & -a-1 & 2a+2 & 0 \\ 0 & 2a+2 & 4a-4 & a^2+a-8 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} a+b \\ 2a+b \\ 0 \\ 4a+ab+b \end{pmatrix}$$

E o sistema linear:

$$(*): \quad \underline{\underline{A}}\,\underline{x} = \underline{b}$$

(a) Encontrar o posto da matriz coefficiente, $\underline{\underline{A}}$, por qualquer valor de $a \in \mathbb{R}$.

(b) Encontrar o posto da matriz augmentada, $\underline{\underline{T}} = (\underline{\underline{A}}|\underline{b})$, por quaisquer valores de $a, b \in \mathbb{R}$.

(c) Encontrar os valores (a, b) tal que o sistema $(*)$ não tem solução.

(d) Encontrar os valores (a, b) tal que o sistema $(*)$ tem solução única.

(e) Encontrar os valores (a, b) tal que o sistema $(*)$ tem infinitas solução.

(f) Resolver o sistema $(*)$ para $(a, b) = (-1, 1)$. Identificar nesta solução a solução completa do sistema homogênea (SCSH) e uma solução particular do sistema inhomogênea (SPSñH).

Solution:

$$\underline{\underline{T}} = (\underline{\underline{A}}|\underline{b}) =$$

$$\left(\begin{array}{cccc|c} 1 & a-1 & 2 & a+2 & a+b \\ 1 & 2a & 0 & a & 2a+b \\ 0 & -a-1 & 2a+2 & 0 & 0 \\ 0 & 2a+2 & 4a-4 & a^2+a-8 & 4a+ab+b \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & a-1 & 2 & a+2 & a+b \\ 0 & a+1 & -2 & -2 & a \\ 0 & -a-1 & 2a+2 & 0 & 0 \\ 0 & 2a+2 & 4a-4 & a^2+a-8 & 4a+ab+b \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & a-1 & 2 & a+2 & a+b \\ 1 & a+1 & -2 & -2 & a \\ 0 & 0 & 2a & -2 & a \\ 0 & 0 & 4a & a^2+a-4 & 2a+ab+b \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & a-1 & 2 & a+2 & a+b \\ 1 & a+1 & -2 & -2 & a \\ 0 & 0 & 2a & -2 & a \\ 0 & 0 & 0 & a(a+1) & b(a+1) \end{array}\right) \sim$$

(a) Vemos que:

$$\rho_{\underline{\underline{A}}} = \left\{ \begin{array}{lll} 2 & por & a = -1 \\ 3 & por & a = 0 \\ 4 & por & a \neq 0 \wedge a \neq -1 \end{array} \right)$$

(b) E:

$$\rho_{\underline{\underline{T}}} = \left\{ \begin{array}{lll} 2 & por & a = -1 \wedge b \in \mathbb{R} \\ 3 & por & a = 0 \wedge b = 0 \\ 4 & por & a = 0 \wedge b \neq 0 \\ 4 & por & a \neq 0 \wedge a \neq -1 \wedge b \in \mathbb{R} \end{array} \right)$$

(c) $(*)$ não tem solução, seeee: $\rho_{\underline{\underline{T}}} > \rho_{\underline{\underline{A}}}$, ou seja: $a = 0 \wedge b \neq 0$.

(d) $(*)$ tem solução único, seeee: $\rho_{\underline{\underline{T}}} = \rho_{\underline{\underline{A}}} = 4$, ou seja: $a \neq 0 \wedge a \neq -1 \wedge b \in \mathbb{R}$.

(e) $(*)$ tem infinitas solução único, seeee: $\rho_{\underline{\underline{T}}} = \rho_{\underline{\underline{A}}} < 4$, ou seja: $a = -1 \wedge b \in \mathbb{R}$ ou: $a = 0 \wedge b = 0$.

(f) No caso $9a, b) = (-1, 1)$:

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} \mathbf{1} & -2 & 2 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 1 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cccc|c} \mathbf{1} & -2 & 0 & -1 & -1 \\ 0 & 0 & \mathbf{1} & 1 & \frac{1}{2} \end{array}\right)$$

Parametrizando: $x_2 = t$ e $x_4 = s$:

$$x_1 = 2x_2 + x_4 - 1 = 2t + s - 1$$

E:

$$x_3 = -x_4 + \frac{1}{2} = -s + \frac{1}{2}$$

Juntando vetorialmente, obtemos a solução completa:

$$\underline{\mathbf{x}} = \begin{pmatrix} -1 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad t, s \in \mathbb{R}$$

2. *3 pts.* Dado os vetores:

$$\underline{\mathbf{a}}_1 = (1, -1, 2, 1)^T \quad \underline{\mathbf{a}}_2 = (0, 1, 1, 3)^T \quad \underline{\mathbf{a}}_3 = (1, -2, 2, -1)^T \quad \underline{\mathbf{a}}_4 = (0, 1, -1, 3)^T \quad \underline{\mathbf{a}}_5 = (1, -2, 2, -3)^T$$

(a) Mostre que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3, \underline{\mathbf{a}}_4$ formam uma base de $\mathbb{R}^4$.

(b) Encontrar os coordenados do vetor $\underline{\mathbf{a}}_5$ neste base.

(c) Encontrar os coordenados dos vetores $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3, \underline{\mathbf{a}}_4$ neste base.

(d) Encontrar os coordenados dos vetores $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3, \underline{\mathbf{a}}_4$ no base canônica.

Solution:

$$\begin{pmatrix} \mathbf{1} & 0 & 1 & 0 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & 1 & 2 & -1 & 2 \\ 1 & 3 & -1 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & \mathbf{1} & 0 & -1 & 0 \\ 0 & 3 & -2 & 3 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -\mathbf{1} & 2 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 6 & -4 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mathbf{2} & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

(a) Do cálculo anterior segue que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3, \underline{\mathbf{a}}_4$ formam uma base de $\mathbb{R}^4$.

(b) Do cálculo anterior também segue que os coordenados do vetor $\underline{\mathbf{a}}_5$ neste base, é: $\underline{\mathbf{a}}_5 = [2, -1, -1, -1]_{\underline{\mathbf{a}}_i} = 2\underline{\mathbf{a}}_1 - \underline{\mathbf{a}}_2 - \underline{\mathbf{a}}_3 - \underline{\mathbf{a}}_4$.

(c) Temos:
$\underline{\mathbf{a}}_1 = [1, 0, 0, 0]^T_{\underline{\mathbf{a}}_i}$
$\underline{\mathbf{a}}_2 = [0, 1, 0, 0]^T_{\underline{\mathbf{a}}_i}$
$\underline{\mathbf{a}}_3 = [0, 0, 1, 0]^T_{\underline{\mathbf{a}}_i}$
$\underline{\mathbf{a}}_4 = [0, 0, 0, 1]^T_{\underline{\mathbf{a}}_i}$

(d) No base canônica:
$\underline{\mathbf{a}}_1 = (1, -1, 2, 1)^T$
$\underline{\mathbf{a}}_2 = (0, 1, 1, 3)^T$
$\underline{\mathbf{a}}_3 = (1, -2, 2, -1)^T$
$\underline{\mathbf{a}}_4 = (0, 1, -1, 3)^T$

3. *3 pts.* Dado os matrices:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix}, \qquad \underline{\underline{\mathbf{B}}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

(a) Mostre que $\underline{\underline{\mathbf{B}}}$ é regular.

(b) Encontrar $\underline{\underline{\mathbf{B}}}^{-1}$.

(c) Resolver a equação matricial: $\underline{\underline{\mathbf{X}}}\,\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{A}}}$.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right) \sim$$
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \sim$$

(a) O cálculo anterior mostre que $\underline{\underline{\mathbf{B}}}$ é regular.

(b) Também segue deste cálculo:

$$\underline{\underline{\mathbf{B}}}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \sim$$

.

(c) Por $\underline{\underline{\mathbf{B}}}$ ser regular, temos:

$$\underline{\underline{\mathbf{X}}}\,\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{A}}} \quad \Leftrightarrow \quad \underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{B}}}^{-1} = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

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Disciplina:	Álgebra Linear
Prova:	I - 2^a Chamada

1. *4 pts.* Dado o matriz, $\underline{\underline{\mathbf{A}}}$, e o vetor, $\underline{\mathbf{b}}$:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & 2a & a \\ 1 & a & 2a & 1 \\ 1 & 1 & a & 2a \\ 1 & a & a & 2a \end{pmatrix}, \qquad \underline{\mathbf{b}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ a \end{pmatrix}$$

E o sistema linear:

$$(*): \quad \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

(a) Encontrar o posto da matriz coefficiente, $\underline{\underline{\mathbf{A}}}$, por qualquer valor de $a \in \mathbb{R}$.

(b) Encontrar o posto da matriz aumentada, $\underline{\underline{\mathbf{T}}} = (\underline{\underline{\mathbf{A}}}|\underline{\mathbf{b}})$, por qualquer valores de $a \in \mathbb{R}$.

(c) Encontrar os valores a tal que o sistema $(*)$ não tem solução.

(d) Encontrar os valores a tal que o sistema $(*)$ tem solução única.

(e) Encontrar os valores a tal que o sistema $(*)$ tem infinitas soluções.

(f) Resolver o sistema $(*)$ para $a = 1$.

(g) Identificar na solução do item anterior a solução completa do sistema homogênea (SCSH) e uma solução particular do sistema inhomogênea (SPSñH).

Solution:

$$\underline{\underline{\mathbf{T}}} = (\underline{\underline{\mathbf{A}}}|\underline{\mathbf{b}}) =$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2a & a & 1 \\ 1 & a & 2a & 1 & 1 \\ 1 & 1 & a & 2a & 1 \\ 1 & a & a & 2a & a \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2a & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & -a & a & 0 \\ 0 & a-1 & -a & a & a-1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2a & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & -a & a & 0 \\ 0 & 0 & -a & 2a-1 & a-1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 2a & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & -a & a & 0 \\ 0 & 0 & 0 & a-1 & a-1 \end{array}\right)$$

(a) Do cálculo anterior:

$$\rho_{\underline{\underline{\mathbf{A}}}} = \begin{cases} 2 & por \quad a = 1 \\ 3 & por \quad a = 0 \\ 4 & por \quad a \neq 0 \wedge a \neq 1 \end{cases}$$

(b) Do cálculo anterior:

$$\rho_{\underline{\underline{\mathbf{T}}}} = \begin{cases} 2 & por \quad a = 1 \\ 3 & por \quad a = 0 \\ 4 & por \quad a \neq 0 \wedge a \neq 1 \end{cases}$$

Ou seja: $\rho_{\underline{\underline{\mathbf{A}}}} = \rho_{\underline{\underline{\mathbf{T}}}}$, $\forall a \in \mathbb{R}$.

(c) Por $\rho_{\underline{\underline{\mathbf{A}}}} = \rho_{\underline{\underline{\mathbf{T}}}}$, $\forall a \in \mathbb{R}$, $(*)$ sempre tem solução.

(d) $(*)$ tem solução única quando: $n = \rho$, ou seja: $a \neq 0 \wedge a \neq 1$.

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(e) $(*)$ tem infinitas soluções quando: $n < \rho$, ou seja: $a = 0 \vee a = 1$.

(f) Colocando $a = 1$ e omitindo linhas com somente zeros:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right)$$

Pondo: $x_2 = t$ e $x_4 = s$: $x_1 = 1 - x_2 - 3x_4 = 1 - t - 3s$, e $x_3 = x_4 = s$. Assim, a solução completa é:

$$\underline{\mathbf{x}} = (1, 0, 0, 0)^T + t(-1, 1, 0, 0)^T + s(-3, 0, 1, 1)^T, \quad t, s \in \mathbb{R}$$

2. *3 pts.* Dado os vetores:

$$\underline{\mathbf{a}}_1 = (1, 0, -1)^T \qquad \underline{\mathbf{a}}_2 = (1, 1, 1)^T \qquad \underline{\mathbf{a}}_3 = (1, -1, 1)^T$$

(a) Mostre que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$ formam uma base de $\mathbb{R}^3$.

(b) Encontrar uma equação expressando coordenados em relação à base $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$, em termos dos coordenados em relação à base canônica em $\mathbb{R}^3$.

(c) Encontrar os coordenados dos vetores básicos da base canônica em $\mathbb{R}^3$, na base $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$.

Solution:

$$(\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{I}}}) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right) = (\underline{\underline{\mathbf{I}}}|\underline{\underline{\mathbf{A}}}^{-1})$$

(a) O cálculo anterior mostre que $\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, \underline{\mathbf{a}}_3$ formam uma base em $\mathbb{R}^3$.

(b) A relação entre coordenados novos e antigos, é: $\underline{\mathbf{x}}_A = \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}}_N$, ou equivalentemente: $\underline{\mathbf{x}}_A = \underline{\underline{\mathbf{A}}}^{-1}\underline{\mathbf{x}}_N$. Pela cálculo anterior:

$$\underline{\mathbf{x}}_N = \underline{\underline{\mathbf{A}}}^{-1}\,\underline{\mathbf{x}}_A = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \underline{\mathbf{x}}_A$$

(c) Os coordenados novos dos vetores básicas antigas, estão nas colunas da inversa:

$$\underline{\mathbf{e}}_1 = (1, 0, 0)^T = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]^T_{\underline{\mathbf{a}}}$$

$$\underline{\mathbf{e}}_2 = (0, 1, 0)^T = \left[0, \frac{1}{2}, -\frac{1}{2}\right]^T_{\underline{\mathbf{a}}}$$

$$\underline{\mathbf{e}}_3 = (0, 0, 1)^T = \left[-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]^T_{\underline{\mathbf{a}}}$$

3. *3 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Mostre que $\underline{\underline{\mathbf{A}}}$ é singular.

(b) Resolver a sistema homogênea: $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

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(c) Resolver a equação matricial: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{A}}}^2$.
Hint: Pode ser conveniente usar, que $\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{A}}}$ é uma solução particular da equação matricial.

Solution:

(a)

$$\det \underline{\underline{\mathbf{A}}} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1 + 1 = 0$$

Ou seja $\underline{\underline{\mathbf{A}}}$ é singular.

(b)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ou seja: $x = y = -z = t$. De forma vetorial: $\underline{\mathbf{x}} = t(1, 1, -1)^t, \;\; t \in \mathbb{R}$.

(c) Pelo item anterior, a solução completa da: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{0}}}$, é:

$$\underline{\underline{\mathbf{X}}} = \begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \\ -t_1 & -t_2 & -t_3 \end{pmatrix}$$

Justificando o *hint*, que $\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{A}}}$ é uma solução particular: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{A}}} = \underline{\underline{\mathbf{A}}}^2$, e aplicando: SCSñH = SPSñH +SCSH, obtemos a solução completa da: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{A}}}^2$x:

$$\underline{\underline{\mathbf{X}}} = \begin{pmatrix} t_1 & 1 + t_2 & 1 + t_3 \\ 1 + t_1 & -1 + t_2 & t_3 \\ 1 - t_1 & -t_2 & 1 - t_3 \end{pmatrix}$$

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Prova: I - 2^a Chamada

1. *4 pts.* Dado as matrizes, $\underline{\underline{\mathbf{A}}}$ e $\underline{\underline{\mathbf{B}}}$:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & a & 1 \\ 2 & a+2 & a-2 \\ 1 & a+1 & a-1 \end{pmatrix}, \qquad \underline{\underline{\mathbf{b}}} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ b & b \\ b-1 & 2b+1 \end{pmatrix}$$

E a equação:

$$(*): \quad \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{B}}}$$

(a) Encontrar o posto da matriz coefficiente, $\underline{\underline{\mathbf{A}}}$, por qualquer valor de $a \in \mathbb{R}$.

(b) Encontrar o posto da matriz augmentada, $\underline{\underline{\mathbf{T}}} = (\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{B}}})$, por quaisquer valores de $(a, b) \in \mathbb{R}^2$.

(c) Encontrar os valores (a, b) tal que a equação $(*)$ não tem solução (incompatível).

(d) Encontrar os valores (a, b) tal que a equação $(*)$ tem solução única (determinado).

(e) Encontrar os valores (a, b) tal que a equação $(*)$ tem infinitas soluções (indeterminado).

(f) Resolver a equação $(*)$ para $(a, b) = (2, 0)$.

Solution:
Calculamos:

$$(\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{B}}}) = \left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 2 & a+2 & a-2 & b & b \\ 1 & a+1 & a-1 & b-1 & 2b+1 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 0 & a & a & b-2 & b+2 \\ 0 & a & a & b-2 & 2b+2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 0 & 0 & a-1 & b & b \\ 0 & 0 & a-1 & b & 2b \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 1 & -1 & 1 & -1 \\ 0 & a & 1 & -2 & 2 \\ 0 & 0 & a-1 & b & b \\ 0 & 0 & 0 & 0 & b \end{array}\right)$$

E a equação matricial:

$$(*): \quad \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{B}}}$$

(a) Vemos:

$$\rho_{\underline{\underline{\mathbf{A}}}} = \begin{cases} 2 & por \quad a = 1 \\ 2 & por \quad a = 0 \\ 3 & por \quad a \neq 0 \wedge a \neq 0 \end{cases}$$

(b) E mais:

$$\rho_{\underline{\underline{\mathbf{T}}}} = \begin{cases} \rho_{\underline{\underline{\mathbf{A}}}} & por \quad b = 0 \\ \rho_{\underline{\underline{\mathbf{A}}}} + 1 & por \quad b \neq 0 \end{cases}$$

(c) Do item (b) segue que $(*)$ não tem solução, seee: $b \neq 0$.

(d) Do item (b) segue que $(*)$ tem solução única, seee: $a \neq 0 \wedge a \neq 1 \wedge b = 0$.

(e) Do item (b) segue que $(*)$ tem infinitas soluções, seee: $(a = 0 \vee a = 1) \wedge b = 0$.

(f) Colocando $(a,b)=(2,0)$ (caso de solução única):

$$\left(\begin{array}{rrr|rr} 1 & 1 & -1 & 1 & -1 \\ 0 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|rr} 1 & 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim$$

$$\left(\begin{array}{rrr|rr} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|rr} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Ou seja:

$$\underline{\underline{\mathbf{X}}} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}$$

2. *3 pts.* Dado as matrizes:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}, \qquad \underline{\underline{\mathbf{B}}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Encontrar $\underline{\underline{\mathbf{A}}}^{-1}$.

(b) Encontrar $\underline{\underline{\mathbf{B}}}^{-1}$.

(c) Encontrar $(\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{B}}})^{-1}$.

Solution:

(a) Calulamos:

$$(\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{I}}}) = \left(\begin{array}{rrr|rrr} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrr|rrr} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) = (\underline{\underline{\mathbf{I}}}|\underline{\underline{\mathbf{A}}}^{-1})$$

(b)

$$(\underline{\underline{\mathbf{B}}}|\underline{\underline{\mathbf{I}}}) = \left(\begin{array}{rrr|rrr} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) = (\underline{\underline{\mathbf{I}}}|\underline{\underline{\mathbf{B}}}^{-1})$$

(c) Temos:

$$(\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{B}}})^{-1} = \underline{\underline{\mathbf{B}}}^{-1}\underline{\underline{\mathbf{A}}}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 2 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

3. *3 pts.* Dado os vetores:

$$\underline{\mathbf{d}}_1 = \frac{1}{2}(1,1,1,1)^T \quad \underline{\mathbf{d}}_2 = \frac{1}{2}(-1,1,-1,1)^T \quad \underline{\mathbf{d}}_3 = \frac{1}{2}(-1,-1,1,1)^T \quad \underline{\mathbf{d}}_4 = \frac{1}{2}(-1,1,1,-1)^T \quad \underline{\mathbf{d}}_5 = (1,2,1,2)^T$$

(a) Mostre que $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$ formam uma base *ortonormal* em $\mathbb{R}^4$.

(b) Encontrar os coordenados do vetor $\underline{\mathbf{d}}_5$ em relação a base $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$.

(c) Encontrar os coordenados dos vetores da base canônica, $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3, \underline{\mathbf{e}}_4)$, em relação a base $(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4)$.

Solution:

(a) Verificamos:

$$\underline{\mathbf{d}}_i \cdot \underline{\mathbf{d}}_j = \delta_{ij} = \begin{cases} 1 & por \quad i = j \\ 0 & por \quad i \neq j \end{cases}$$

Ou seja $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4$ são ortonormais. Por serem mutualmente ortogonais, $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4$ são linearmente independentes, e assim formam uma base em $\mathbb{R}^4$.

Por $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3, \underline{\mathbf{d}}_4$ serem ortonormais, temos: $\underline{\underline{\mathbf{D}}}^{-1} = \underline{\underline{\mathbf{D}}}^T$. As relações entre os coordenados novos, $\underline{\mathbf{x}}'$, e os coordenados antigos, $\underline{\mathbf{x}}$, são:

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{D}}}\,\underline{\mathbf{x}}' = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\underline{\mathbf{x}}' \quad \Leftrightarrow \quad \underline{\mathbf{x}}' = \underline{\underline{\mathbf{D}}}^{-1}\underline{\mathbf{x}} = \underline{\underline{\mathbf{D}}}^T\underline{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}\underline{\mathbf{x}}$$

(b)

$$\underline{\mathbf{d}}_5' = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) Os coordenados dos vetores da base canônica estão nas colunas do inverso:

$$\underline{\mathbf{e}}_1' = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad \underline{\mathbf{e}}_2' = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{e}}_3' = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{e}}_3' = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

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1. *4 pts.* Dado a forma quadrática:

$$F_2(x,y,z) = 6y^2 + 12xz$$

(a) Encontrar uma matriz simétrica, tal que: $F(\underline{\mathbf{x}}) = \underline{\mathbf{x}}^T \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{x}}$, onde $\underline{\mathbf{x}} = (x,y,z)$.

(b) Encontrar os autovalores do $\underline{\underline{\mathbf{A}}}$.

(c) Encontrar os autovetores do $\underline{\underline{\mathbf{A}}}$.

(d) Encontrar uma base ortonormal de autovetores do $\underline{\underline{\mathbf{A}}}$.

(e) Encontrar uma matriz ortogonal,$\underline{\underline{\mathbf{D}}}$, e uma matriz diagonal, $\underline{\underline{\mathbf{B}}}$, tal que: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{D}}}^{-1} \underline{\underline{\mathbf{A}}}\, \underline{\underline{\mathbf{D}}}$.

(f) Com este substituição ortogonal, encontre uma relação entre os coordenados novos, $\underline{\mathbf{x}}'$, e os coordenados antigos, $\underline{\mathbf{x}}$, e vice-versa.

(g) Encontrar $F_2'(\underline{\mathbf{x}}') = F_2(\underline{\mathbf{x}})$.

(h) Classifique a superfície:

$$6y^2 + 12xz + 2x - 2y + 2z = 3$$

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 6 & 0 \\ 6 & 0 & 0 \end{pmatrix}$$

E: $F_2(x,y,x) = \underline{\mathbf{x}}^T \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{x}}$.

(b)

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda \underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} -\lambda & 0 & 6 \\ 0 & 6-\lambda & 0 \\ 6 & 0 & -\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} -\lambda & 6 \\ 6 & -\lambda \end{vmatrix} = (6-\lambda)(\lambda^2 - 36) = 0 \Leftrightarrow$$

$$\lambda = 6 \vee \lambda = -6$$

(c) $\lambda = 6$:

$$\underline{\underline{\mathbf{A}}} - 6\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -6 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow$$

$$-x_1 + x_3 = 0 \Leftrightarrow \underline{\mathbf{x}} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t,s \in \mathbb{R}$$

$\lambda = -6$: Os autovetores são ortogonais aos dois autovetores de $\lambda = 6$, ou seja:

$$\underline{\mathbf{x}} = t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(d) Os autovetores do $\lambda = 6$ já são ortognais e ambos ortogonais ao autovetor do $\lambda = 6$. Normalizando:

$$\underline{\mathbf{v}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{v}}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

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(e) Colocando os $\underline{\mathbf{v}}_i$ em colunas:

$$\underline{\underline{\mathbf{D}}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \underline{\underline{\mathbf{D}}}^T$$

E:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(f)

$$\underline{\mathbf{x}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \underline{\mathbf{x}}' \quad \Leftrightarrow \quad \underline{\mathbf{x}}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \underline{\mathbf{x}}$$

(g) $F_2'(\underline{\mathbf{x}}') = 6x_1^2 + 6y_1^2 - 6z_1^2$.

(h) Pelos items anteriores:

$$x = \frac{x_1 + z_1}{\sqrt{2}}$$
$$y = y_1$$
$$z = \frac{x_1 - z_1}{\sqrt{2}}$$

Assim:

$$2x - 2y + 2z = \sqrt{2}x_1 - 2y_1$$

A forma quadrática se transforma em:

$$6x_1^2 + 6y_1^2 - 6z_1^2 + \sqrt{2}x_1 - 2y_1 = 3 \Leftrightarrow$$
$$6\left(x_1^2 + \frac{\sqrt{2}}{6}x_1\right) + 6\left(y_1^2 - \frac{1}{3}y_1\right) - 6z_1^2 = 3 \Leftrightarrow$$
$$6\left(x_1 + \frac{\sqrt{2}}{12}\right)^2 - \frac{1}{12} + 6\left(y_1 - \frac{1}{6}\right)^2 - \frac{1}{6} - 6z_1^2 = 3 \Leftrightarrow$$
$$6\left(x_1 + \frac{\sqrt{2}}{12}\right)^2 + 6\left(y_1 - \frac{1}{6}\right)^2 - 6z_1^2 = 3 + \frac{1}{12} + \frac{1}{6} = \frac{39}{12} \Leftrightarrow$$
$$\frac{\left(x_1 + \frac{\sqrt{2}}{12}\right)^2}{\left(\sqrt{\frac{39}{72}}\right)^2} + \frac{\left(y_1 - \frac{1}{6}\right)^2}{\left(\sqrt{\frac{39}{72}}\right)^2} - \frac{z_1^2}{\left(\sqrt{\frac{39}{72}}\right)^2} = 1$$

Este superfície é uma hiperbolóide de revolução de uma folha. Semi-eixos: $a = b = c = \frac{39}{72}$, e centro: $(x_1, y_1, z_1) = (-\frac{\sqrt{2}}{12}, \frac{1}{6}, 0)$, ou: $(x, y, z) = (-\frac{1}{12}, \frac{1}{6}, -\frac{1}{12})$. Eixo de revolução paralelo com o eixo z_1, ou seja: $(1, 0, -1)^T$. Parametrização:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{12}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$$

2. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

E a aplicação linear: $f : \mathbb{R}^3 \mapsto \mathbb{R}^3 : f(\underline{\mathbf{x}}) = \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{x}}$.

(a) Encontrar o núcleo de f e sua dimensão.

(b) Encontrar a dimensão e uma base da imagem da f.

(c) Pondo, $\underline{\mathbf{d}} = (1, 2, 1)^T$, encontrar a solução completa de: $f(\underline{\mathbf{x}}) = f(\underline{\mathbf{d}})$.

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 9 & 0 \\ -1 & 5 & -1 \\ 0 & 9 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}}$$

Temos: $\rho = 2$, assim a dimensão do núcleo é $3 - 2 = 1$, e:

$$x_1 = -x_3 \wedge x_2 = 0 \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(b) A dimensão da imagem é $\rho = 2$. Uma base da imagem seria dois vetores LI das colunas da $\underline{\underline{\mathbf{A}}}$, por exemplo:

$$\underline{\mathbf{v}}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad \underline{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$$

(c) Obviamente $\underline{\mathbf{d}}$ é uma solução particular da equação. Obtemos a solução completa adicionando a solução completa da equção homogênea, ou seja: $\underline{\mathbf{x}} = \underline{\mathbf{d}} + t(-1, 0, 1)^T$, $t \in \mathbb{R}$.

3. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 10 & -2\sqrt{3} \\ -2\sqrt{3} & 6 \end{pmatrix}$$

E uma função bilinear: $g(x, y) = (x\ y)\underline{\underline{\mathbf{A}}} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Encontrar os autovalores e autovetores do $\underline{\underline{\mathbf{A}}}$.

(b) Mostre que $g(x, y)$ define um produto interno em $\mathbb{R}^2$.

(c) Dado o vetor $\underline{\mathbf{v}}_1 = (1, -1)^T$, encontrar um vetor, $\underline{\mathbf{v}}_2$ ortogonal ao $\underline{\mathbf{v}}_1$ ao respeito de g.

(d) Encontrar uma base ortonormal ao respeito do produto interno g.

Solution:

(a)

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda\underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} 10 - \lambda & -2\sqrt{3} \\ -2\sqrt{3} & 6 - \lambda \end{vmatrix} = (10 - \lambda)(6 - \lambda) - 12 = 48 - 16\lambda - \lambda^2 = 0 \Leftrightarrow \lambda = 12 \vee \lambda = 4$$

$\lambda = 4$:

$$\underline{\underline{\mathbf{A}}} - 4\underline{\underline{\mathbf{I}}} = \begin{pmatrix} 6 & -2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix} \sim \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad t \in \mathbb{R}$$

O autovetor do $\lambda = 12$ é ortogonal e á o autovetor do $\lambda = 4$:

$$\underline{\mathbf{x}} = t \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(b) $g(x, y)$ é claramente bilinear. É simétrica também porque $\underline{\underline{\mathbf{A}}}$ é. Finalmente e positivamente definido, pois ambos aautovalores são positivas.

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(c) Calculamos:

$$\underline{\underline{\mathbf{A}}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 10+2\sqrt{3} \\ -6-2\sqrt{3} \end{pmatrix}$$

O versor deste é orotognal ao $(1,-1)^T$ ao respeito do g:

$$\begin{pmatrix} 6+2\sqrt{3} \\ 10+2\sqrt{3} \end{pmatrix}$$

(d) Podiamos normalizar (ao resp. do g) os vetores no item anterior, porém e mais facil normalizar os autovetores do primeiro item, dividindo por seu complemento vezes o raíz do autovalor. Os autovetores normalizados ao respeito do produto escalar normal:

$$\underline{\mathbf{v}}_1 = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

Dividindo com os $\sqrt{12}$ resp. 2: :

$$\underline{\mathbf{v}}_1 = \frac{1}{2\sqrt{12}}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

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Prova:	II - 1^a Chamada

1. *4 pts.* Dado a forma quadrática:

$$F_2(x,y,z) = 4z^2 + 8xy$$

(a) Encontrar uma matriz simétrica, tal que: $F(\underline{\mathbf{x}}) = \underline{\mathbf{x}}^T \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{x}}$, onde $\underline{\mathbf{x}} = (x,y,z)$.

(b) Encontrar os autovalores do $\underline{\underline{\mathbf{A}}}$.

(c) Encontrar os autovetores do $\underline{\underline{\mathbf{A}}}$.

(d) Encontrar uma base ortonormal de autovetores do $\underline{\underline{\mathbf{A}}}$.

(e) Encontrar uma matriz ortogonal,$\underline{\underline{\mathbf{D}}}$, e uma matriz diagonal, $\underline{\underline{\mathbf{B}}}$, tal que: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{D}}}^{-1} \underline{\underline{\mathbf{A}}}\, \underline{\underline{\mathbf{D}}}$.

(f) Com este substituição ortogonal, encontre uma relação entre os coordenados novos, $\underline{\mathbf{x}}'$, e os coordenados antigos, $\underline{\mathbf{x}}$, e vice-versa.

(g) Encontrar $F_2'(\underline{\mathbf{x}}') = F_2(\underline{\mathbf{x}})$.

(h) Classifique a superfície:

$$4z^2 + 8xy + 2x - 2y = 3$$

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

E: $F_2(x,y,x) = \underline{\mathbf{x}}^T \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{x}}$.

(b)

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda \underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} -\lambda & 4 & 0 \\ 4 & -\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} -\lambda & 4 \\ 4 & -\lambda \end{vmatrix} = (4-\lambda)(\lambda^2 - 16) = 0 \Leftrightarrow$$

$$\lambda = 4 \vee \lambda = -4$$

(c) $\lambda = 4$:

$$\underline{\underline{\mathbf{A}}} - 4\underline{\underline{\mathbf{I}}} = \begin{pmatrix} -4 & 4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow$$

$$-x_1 + x_2 = 0 \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

$\lambda = -4$: Os autovetores são ortogonais aos dois autovetores de $\lambda = 4$, ou seja:

$$\underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

(d) Os autovetores do $\lambda = 4$ já são ortognais e ambos ortogonais ao autovetor do $\lambda = 4$. Normalizando:

$$\underline{\mathbf{v}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(e) Colocando os $\underline{\mathbf{v}}_i$ em colunas:

$$\underline{\underline{\mathbf{D}}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

E:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

(f)

$$\underline{\mathbf{x}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \underline{\mathbf{x}}' \quad \Leftrightarrow \quad \underline{\mathbf{x}}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \underline{\mathbf{x}}$$

(g) $F_2'(\underline{\mathbf{x}}') = 4x_1^2 + 4y_1^2 - 4z_1^2$.

(h) Pelos items anteriores:

$$x = \frac{x_1 - z_1}{\sqrt{2}}$$
$$y = \frac{x_1 + z_1}{\sqrt{2}}$$
$$z = y_1$$

Assim:

$$2x - 2y = -\sqrt{2}z_1$$

A forma quadrática se transforma em:

$$4x_1^2 + 4y_1^2 - 4z_1^2 - \sqrt{2}z_1 = 3 \Leftrightarrow$$
$$x_1^2 + y_1^2 - z_1^2 - \frac{\sqrt{2}}{4}z_1 = \frac{3}{4} \Leftrightarrow$$
$$x_1^2 + y_1^2 - \left[z_1^2 + \frac{\sqrt{2}}{4}z_1\right] = \frac{3}{4} \Leftrightarrow$$
$$x_1^2 + y_1^2 - \left[\left(z_1 + \frac{\sqrt{2}}{8}\right)^2 - \frac{1}{32}\right] = \frac{3}{4} \Leftrightarrow$$
$$x_1^2 + y_1^2 - \left(z_1 + \frac{\sqrt{2}}{8})^2\right) = \frac{3}{4} - \frac{1}{32} = \frac{23}{32}$$

Este superfície é uma hiperbolóide de revolução de uma folha. Semi-eixos: $a = b = c = \frac{23}{32}$, e centro: $(x_1, y_1, z_1) = (0, 0, -\frac{\sqrt{2}}{8})$, ou: $(x, y, z) = (\frac{1}{8}, -\frac{1}{8}, 0)$. Eixo de revolução paralelo com o eixo z_1, ou seja: $(-1, 1, 0)^T$. Parametrização:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

2. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

E a aplicação linear: $f : \mathbb{R}^3 \mapsto \mathbb{R}^3 : f(\underline{\mathbf{x}}) = \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}}$.

(a) Encontrar o núcleo de f e sua dimensão.

(b) Encontrar a dimensão e uma base da imagem da f.

(c) Pondo, $\underline{\mathbf{d}} = (1, -1, 0)^T$, encontrar a solução completa de: $f(\underline{\mathbf{x}}) = f(\underline{\mathbf{d}})$.

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}}$$

Temos: $\rho = 2$, assim a dimensão do núcleo é $3 - 2 = 1$, e:

$$x_1 = x_3 \wedge x_2 = -2x_3 \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(b) A dimensão da imagem é $\rho = 2$. Uma base da imagem seria dois vetores LI das colunas da $\underline{\underline{\mathbf{A}}}$, por exemplo:

$$\underline{\mathbf{v}}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \underline{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(c) Obviamente $\underline{\mathbf{d}}$ é uma solução particular da equação. Obtemos a solução completa adicionando a solução completa da equção homogênea, ou seja: $\underline{\mathbf{x}} = \underline{\mathbf{d}} + t(1, -2, 1)^T$, $t \in \mathbb{R}$.

3. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{pmatrix}$$

E uma função bilinear: $g(x, y) = (x\ y)\underline{\underline{\mathbf{A}}} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Encontrar os autovalores e autovetores do $\underline{\underline{\mathbf{A}}}$.

(b) Mostre que $g(x, y)$ define um produto interno em $\mathbb{R}^2$.

(c) Dado o vetor $\underline{\mathbf{v}}_1 = (1, -1)^T$, encontrar um vetor, $\underline{\mathbf{v}}_2$ ortogonal ao $\underline{\mathbf{v}}_1$ ao respeito de g.

(d) Encontrar uma base ortonormal ao respeito do produto interno g.

Solution:

(a)

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda \underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} 7-\lambda & -\sqrt{3} \\ -\sqrt{3} & 5-\lambda \end{vmatrix} = (7-\lambda)(5-\lambda) - 3 = 32 - 12\lambda + \lambda^2 = 0 \Leftrightarrow \lambda = 8 \vee \lambda = 4$$

$\lambda = 4$:

$$\underline{\underline{\mathbf{A}}} - 4\underline{\underline{\mathbf{I}}} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad t \in \mathbb{R}$$

O autovetor do $\lambda = 8$ é ortogonal no autovetor do $\lambda = 4$:

$$\underline{\mathbf{x}} = t \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(b) $g(x, y)$ é claramente bilinear. É simétrica também porque $\underline{\underline{\mathbf{A}}}$ é. Finalmente e positivamente definido, pois ambos aautovalores são positivas.

(c) Calculamos:

$$\underline{\underline{\mathbf{A}}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7+\sqrt{3} \\ -5-\sqrt{3} \end{pmatrix}$$

O versor deste é orotognal ao $(1,-1)^T$ ao respeito do g:

$$\begin{pmatrix} 5+\sqrt{3} \\ 7+\sqrt{3} \end{pmatrix}$$

(d) Podiamos normalizar (ao resp. do g) os vetores no item anterior, porém e mais facil normalizar os autovetores do primeiro item, dividindo por seu complemento vezes o raíz do autovalor. Os autovetores normalizados ao respeito do produto escalar normal:

$$\underline{\mathbf{v}}_1 = \frac{1}{2} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

Dividindo com os $\sqrt{8}$ resp. 2:

$$\underline{\mathbf{v}}_1 = \frac{1}{2\sqrt{8}} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad \underline{\mathbf{v}}_2 = \frac{1}{4} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

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Prova:	II - 2^a Chamada

1. *3 pts.* Dado os vetores:

$$\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Uma aplicação linear, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ é dado por:

$$f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_3 \quad f(\underline{\mathbf{v}}_2) = \underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_3 \quad f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_1$$

(a) Mostre que os vetores $\underline{\mathbf{v}}_i$ formam uma base em $\mathbb{R}^3$.

(b) Encontrar o matriz de f na base $\underline{\mathbf{v}}_i$.

(c) Encontrar o matriz de f na base canônica, $\underline{\mathbf{e}}_i$.

Solution:

(a)

$$\left(\underline{\underline{\mathbf{V}}} \,\middle|\, \underline{\underline{\mathbf{I}}}\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) = \left(\underline{\underline{\mathbf{I}}} \,\middle|\, \underline{\underline{\mathbf{V}}}^{-1}\right)$$

Mostrando que os $\underline{\mathbf{v}}_i$ formam uma base de $\mathbb{R}^3$.

(b) Colocamos as imagens dos vetores básicos nas coluns do $\underline{\underline{\mathbf{B}}}$:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

(c) Temos: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{V}}}^{-1}\underline{\underline{\mathbf{A}}}\underline{\underline{\mathbf{V}}} \quad \Leftrightarrow \quad \underline{\underline{\mathbf{A}}} = \underline{\underline{\mathbf{V}}}\underline{\underline{\mathbf{B}}}\underline{\underline{\mathbf{V}}}^{-1}$:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 2 \end{pmatrix}$$

E a aplicação linear: $f : \mathbb{R}^3 \mapsto \mathbb{R}^3 : f(\underline{\mathbf{x}}) = \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}}$.

(a) Encontrar autovalores e autovetores do $\underline{\underline{\mathbf{A}}}$.

(b) Encontrar uma base ortonormal de autovetores da $\underline{\underline{\mathbf{A}}}$ e o matriz de f neste base.

Solution:

Gosto tanto do seu Cristo, mas tão pouco dos seus Cristões
Por que seus Cristões não podem ser mais como seu Cristo?
Mahatma Gandhi

(a)

$$\underline{\underline{\mathbf{A}}}-\lambda\underline{\underline{\mathbf{I}}}=\begin{vmatrix}2-\lambda & -2 & 1\\ -2 & 5-\lambda & -2\\ 1 & -2 & 2-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda & -2 & 1\\ 0 & 1-\lambda & 2-2\lambda\\ 1 & -2 & 2-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}2-\lambda & -2 & 1\\ 0 & 1 & 2\\ 1 & -2 & 2-\lambda\end{vmatrix}=$$

$$(1-\lambda)\begin{vmatrix}2-\lambda & -2 & 5\\ 0 & 1 & 0\\ 1 & -2 & 6-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}2-\lambda & 5\\ 1 & 6-\lambda\end{vmatrix}=(1-\lambda)(\lambda^2-8\lambda+7)=0\ \Leftrightarrow\ \lambda=1\vee\lambda=7$$

Onde $\lambda=7$ é raíz simples e $\lambda=1$ é raíz simples.
$\lambda=1$:

$$\underline{\underline{\mathbf{A}}}-\underline{\underline{\mathbf{I}}}=\begin{pmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{pmatrix}\sim\begin{pmatrix}1 & -2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\underline{\mathbf{x}}=\underline{\mathbf{0}}\ \Leftrightarrow\ x_1-2x_2+x_3=0\ \Leftrightarrow$$

$$x_2=t\wedge x_3=s\wedge x_1=2t-s\ \Leftrightarrow\ \underline{\mathbf{x}}=t\begin{pmatrix}2\\1\\0\end{pmatrix}+s\begin{pmatrix}-1\\0\\1\end{pmatrix},\ \ t,s\in\mathbb{R}$$

Por $\underline{\underline{\mathbf{A}}}$ ser simétrica o autovetor do raíz simples, $\lambda=7$ é ortogonal do ambos autovetores acima, ou seja:

$$\underline{\mathbf{x}}=t\begin{pmatrix}1\\-2\\1\end{pmatrix},\ \ t\in\mathbb{R}$$

(b) Escolhemos:

$$\underline{\mathbf{v}}_1=\begin{pmatrix}1\\-2\\1\end{pmatrix}\ \sim\ \lambda=7$$

E:

$$\underline{\mathbf{v}}_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}\ \sim\ \lambda=1$$

E:

$$\underline{\mathbf{v}}_3\parallel\underline{\mathbf{v}}_1\times\underline{\mathbf{v}}_2\parallel\begin{pmatrix}1\\1\\-1\end{pmatrix}\ \sim\ \lambda=1$$

Normalizando:

$$\underline{\underline{\mathbf{D}}}=\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

E:

$$\underline{\underline{\mathbf{B}}}=\begin{pmatrix}7 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

3. *3 pts.* Dado a forma quadrática:

$$F(x,y)=7x^2-2\sqrt{3}xy+5y^2$$

(a) Encontrar uma matriz simétrica, $\underline{\underline{\mathbf{A}}}$, tal que: $F(x,y)=(x\ y)\,\underline{\underline{\mathbf{A}}}\begin{pmatrix}x\\y\end{pmatrix}$.

(b) Encontrar autovetores e autovalores da $\underline{\underline{\mathbf{A}}}$.

(c) Encontrar uma substituição ortogonal, $\underline{\underline{\mathbf{D}}}$, que transforma F em uma forma sem o termo xy.

(d) Classificar a curva: $7x^2 - 2\sqrt{3}xy + 5y^2 + x = 4$.

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 7 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{pmatrix}$$

(b)

$$\det\left(\underline{\underline{\mathbf{A}}} - \lambda\underline{\underline{\mathbf{I}}}\right) = \begin{vmatrix} 7-\lambda & -\sqrt{3} \\ -\sqrt{3} & 5-\lambda \end{vmatrix} = (7-\lambda)(5-\lambda) - 3 = 32 - 12\lambda + \lambda^2 = 0 \Leftrightarrow \lambda = 8 \vee \lambda = 4$$

$\lambda = 4$:

$$\underline{\underline{\mathbf{A}}} - 4\underline{\underline{\mathbf{I}}} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow \underline{\mathbf{x}} = t\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad t \in \mathbb{R}$$

O autovetor do $\lambda = 8$ é ortogonal no autovetor do $\lambda = 4$:

$$\underline{\mathbf{x}} = t\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(c) Substituição ortogonal:

$$\underline{\underline{\mathbf{D}}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

E:

$$F(x,y) = 7x^2 - 2\sqrt{3}xy + 5y^2 = 4x'^2 + 8y'^2$$

(d)

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{D}}}\underline{\mathbf{x}}' \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \underline{\mathbf{x}}' = \begin{pmatrix} \frac{x'-\sqrt{3}y'}{2} \\ \frac{\sqrt{3}x'+y'}{2} \end{pmatrix}$$

$$7x^2 - 2\sqrt{3}xy + 5y^2 + x = 4 \quad \Leftrightarrow$$

$$4x'^2 + 8y'^2 + \frac{x' - \sqrt{3}y'}{2} = 4 \quad \Leftrightarrow$$

$$4\left[x'^2 + \frac{x'}{8}\right] + 8\left[y'^2 - \frac{\sqrt{3}y'}{16}\right] = 4 \quad \Leftrightarrow$$

$$4\left[\left(x' + \frac{1}{16}\right)^2 - \frac{1}{64}\right] + 8\left[\left(y' - \frac{\sqrt{3}}{32}\right)^2 - \frac{3}{256}\right] = 4 \quad \Leftrightarrow$$

$$4\left[\left(x' + \frac{1}{16}\right)^2\right] - \frac{1}{16} + 8\left[\left(y' - \frac{\sqrt{3}}{32}\right)^2\right] - \frac{3}{32} = 4 \quad \Leftrightarrow$$

$$4\left(x' + \frac{1}{16}\right)^2 + 8\left(y' - \frac{\sqrt{3}}{32}\right)^2 = \frac{133}{32} \quad \Leftrightarrow$$

$$\frac{(x' + \frac{1}{16})^2}{\left(\sqrt{\frac{133}{128}}\right)^2} + \frac{\left(y' - \frac{\sqrt{3}}{32}\right)^2}{\left(\sqrt{\frac{133}{256}}\right)^2} = 1$$

Isto é um elipse com semieixos $a = \sqrt{\frac{133}{128}}$ e $b = \sqrt{\frac{133}{256}}$ e centro: $(x', y') = (\frac{-1}{16}, \frac{\sqrt{3}}{32})$. Transformando: $(x, y) = (\frac{1}{32}, 0)$.