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Data:	14/05/2010
Semestre:	2010.1
Curso:	Engenharia de Alimentos
Disciplina:	Álgebra Linear
Prova:	II

1. *4 pts.* Dados os vetores em $\mathbb{R}^4$:

$$\underline{\mathbf{d}}_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\quad \underline{\mathbf{d}}_2=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\quad \underline{\mathbf{d}}_3=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\quad \underline{\mathbf{d}}_4=\begin{pmatrix}1\\0\\0\\2\end{pmatrix}\quad \underline{\mathbf{d}}_5=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

   (a) Mostre que os vetores $\underline{\mathbf{d}}_1,\underline{\mathbf{d}}_2,\underline{\mathbf{d}}_3,\underline{\mathbf{d}}_4$ formam uma base em $\mathbb{R}^4$.

   (b) Encontre uma relação entre as coordenadas em relao a base cannica e a base $(\underline{\mathbf{d}}_1,\underline{\mathbf{d}}_2,\underline{\mathbf{d}}_3,\underline{\mathbf{d}}_4)$.

   (c) Encontre as coordenadas do vetor $\underline{\mathbf{d}}_5$ na base $(\underline{\mathbf{d}}_1,\underline{\mathbf{d}}_2,\underline{\mathbf{d}}_3,\underline{\mathbf{d}}_4)$.

   (d) Encontre as coordenadas antigas dos vetores bsicos novos.

   (e) Encontre as coordenadas novas dos vetores bsicos antigos.

Solution:

(a)

$$\left(\underline{\underline{\mathbf{D}}}\,\middle|\,\underline{\underline{\mathbf{I}}}\right)=\left(\begin{array}{cccc|cccc}\mathbf{1}&0&0&1&1&0&0&0\\1&1&0&0&0&1&0&0\\0&1&1&0&0&0&1&0\\0&0&1&2&0&0&0&1\end{array}\right)\sim\left(\begin{array}{cccc|cccc}1&0&0&1&1&0&0&0\\0&\mathbf{1}&0&-1&-1&1&0&0\\0&1&1&0&0&0&1&0\\0&0&1&2&0&0&0&1\end{array}\right)\sim$$

$$\left(\begin{array}{cccc|cccc}1&0&0&1&1&0&0&0\\0&1&0&-1&-1&1&0&0\\0&0&\mathbf{1}&1&1&-1&1&0\\0&0&1&2&0&0&0&1\end{array}\right)\sim\left(\begin{array}{cccc|cccc}1&0&0&1&1&0&0&0\\0&1&0&-1&-1&1&0&0\\0&0&1&1&1&-1&1&0\\0&0&0&\mathbf{1}&-1&1&-1&1\end{array}\right)\sim$$

$$\left(\begin{array}{cccc|cccc}1&0&0&0&2&-1&1&-1\\0&1&0&0&-2&2&-1&1\\0&0&1&0&2&-2&2&-1\\0&0&0&1&-1&1&-1&1\end{array}\right)$$

(b) Temos: $\underline{\mathbf{x}}_A=\underline{\underline{\mathbf{D}}}\,\underline{\mathbf{x}}_N\Leftrightarrow\underline{\mathbf{x}}_N=\underline{\underline{\mathbf{D}}}^{-1}\underline{\mathbf{x}}_A$, por completo:

$$\underline{\mathbf{x}}_A=\begin{pmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&2\end{pmatrix}\underline{\mathbf{x}}_N\Leftrightarrow\underline{\mathbf{x}}_N=\begin{pmatrix}2&-1&1&-1\\-2&2&-1&1\\2&-2&2&-1\\-1&1&-1&1\end{pmatrix}\underline{\mathbf{x}}_A$$

(c) Em relao a base $(\underline{\mathbf{d}}_1,\underline{\mathbf{d}}_2,\underline{\mathbf{d}}_3,\underline{\mathbf{d}}_4)$:

$$\underline{\mathbf{d}}_5=\begin{pmatrix}2&-1&1&-1\\-2&2&-1&1\\2&-2&2&-1\\-1&1&-1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\underline{\mathbf{d}}_1+\underline{\mathbf{d}}_3$$

(d) As coordenadas antigas dos vetores bsicos novos estão nas colunas da $\underline{\underline{\mathbf{D}}}$:

$$\underline{\mathbf{d}}_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\quad \underline{\mathbf{d}}_2=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\quad \underline{\mathbf{d}}_3=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\quad \underline{\mathbf{d}}_4=\begin{pmatrix}1\\0\\0\\2\end{pmatrix}$$

(e) As coordenadas novas dos vetores bsicos antigos estão nas colunas da $\underline{\underline{\mathbf{D}}}^{-1}$:

$$\underline{\mathbf{e}}_1 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix} \quad \underline{\mathbf{e}}_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \quad \underline{\mathbf{e}}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} \quad \underline{\mathbf{e}}_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

2. *2 pts.* Uma aplicao linear, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ dado pela sua matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Encontrar o núcleo e a sua dimensão.

(b) Encontrar a dimenso e uma base da imagem, $f(\mathbb{R}^3)$.

(c) Encontrar o conjunto: $\{\underline{\mathbf{x}} \in \mathbb{R}^3 \mid f(\underline{\mathbf{x}}) = (0,1,1)\}$.

(d) f tem inversa?

Solution:

(a)

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 1 & 1 \\ \mathbf{1} & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & \mathbf{1} & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow$$

$$x_2 + x_3 = 0 \wedge x_1 + x_3 = 0 \Leftrightarrow x_3 = t \wedge x_1 = x_2 = -t \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

A dimenso do ncleo $n - \rho_{\underline{\underline{\mathbf{A}}}} = 3 - 2 = 1$.

(b) A dimenso da imagem $\rho_{\underline{\underline{\mathbf{A}}}} = 2$. Para encontrar uma base desta, selecionamos 2 vetores linearmente independentes das colunas do $\underline{\underline{\mathbf{A}}}$, por exemplo: $\underline{\mathbf{a}}_1 = (0,1,1)$ e $\underline{\mathbf{a}}_2 = (1,-1,0)$.

(c) Observamos que $f(\underline{\mathbf{i}}) = (0,1,1)$, ou seja: $\underline{\mathbf{x}} = \underline{\mathbf{i}}$ uma soluo *particular* do sistema homogłnea $f(\underline{\mathbf{x}} = (0,1,1)$. Obtemos a soluo completa deste, adicionando a soluo completa do sistema homegłnea (o ncleo):

$$f(\underline{\mathbf{x}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow \underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(d) Por $\rho_{\underline{\underline{\mathbf{A}}}} = 2 < 3$, $\underline{\underline{\mathbf{A}}}$ *não* é regular, assim f *não* tem inversa.

3. *4 pts.* Dados os vetores em $\mathbb{R}^4$:

$$\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix} \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix} \quad \underline{\mathbf{v}}_3 = \begin{pmatrix} -1 \\ 1 \\ -3 \\ 5 \end{pmatrix} \quad \underline{\mathbf{v}}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

e uma aplicao linear, $f : \mathbb{R}^4 \mapsto \mathbb{R}^4$:

$$f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2, \quad f(\underline{\mathbf{v}}_2) = -\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2, \quad f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_3 + \underline{\mathbf{v}}_4, \quad f(\underline{\mathbf{v}}_4) = -\underline{\mathbf{v}}_3 + \underline{\mathbf{v}}_4$$

(a) Mostrar que os vetores $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4$ formam uma base em $\mathbb{R}^4$.

(b) Encontrar a matriz, $\underline{\underline{\mathbf{A}}}$, em relao a base canônica (no domínio e na imagem).

(c) Encontrar a matriz, $\underline{\underline{\mathbf{B}}}$, em relao da base $(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4)$ (no domínio e na imagem).

(d) Sendo $U = ger(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2)$, mostre que $f(U) = U$.

(e) Sendo $\underline{\underline{\mathbf{V}}}$ a matriz contendo os vetores $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4$ em colunas, mostre: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{V}}}^{-1} \underline{\underline{\mathbf{A}}}\, \underline{\underline{\mathbf{V}}}$.

Solution:

(a) Mostramos que os vetores formam uma base e encontramos a inversa:

$$\left(\underline{\underline{\mathbf{V}}} \,\middle|\, \underline{\underline{\mathbf{I}}}\right) = \left(\begin{array}{rrrr|rrrr} \mathbf{1} & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 3 & -3 & -3 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrrr|rrrr} 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 2 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 2 & -2 & 1 & 0 & 0 \\ 0 & \mathbf{1} & -1 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & 0 & -1 & 10 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 & -16 & -3 & 6 & 1 \end{array}\right) \sim \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & 0 & -1 & 10 & 1 & -3 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 16 & 3 & -6 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 6 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 26 & 4 & -9 & -1 \\ 0 & 0 & 0 & 1 & 16 & 3 & -6 & -1 \end{array}\right) \sim \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 26 & 4 & -9 & -1 \\ 0 & 0 & 1 & 0 & 6 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 16 & 3 & -6 & -1 \end{array}\right)$$

Este clculo mostra que o matriz $\underline{\underline{\mathbf{V}}}$ regular, ou seja que $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4$ formam uma base em $\mathbb{R}^4$.

(b) Encontramos as imagens:

$$f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 7 \\ -4 \end{pmatrix}$$

$$f(\underline{\mathbf{v}}_2) = -\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2 = -\begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \end{pmatrix}$$

$$f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_3 + \underline{\mathbf{v}}_4 = \begin{pmatrix} -1 \\ 1 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -6 \\ 6 \end{pmatrix}$$

$$f(\underline{\mathbf{v}}_4) = -\underline{\mathbf{v}}_3 + \underline{\mathbf{v}}_4 = -\begin{pmatrix} -1 \\ 1 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}$$

Isto mostre, que: $\underline{\underline{\mathbf{A}}}\, \underline{\mathbf{v}}_1 = (2,2,7,-4)^T$, $\underline{\underline{\mathbf{A}}}\, \underline{\mathbf{v}}_2 = (0,-2,-1,0)^T$, $\underline{\underline{\mathbf{A}}}\, \underline{\mathbf{v}}_3 = (-2,1,-6,6)^T$, $\underline{\underline{\mathbf{A}}}\, \underline{\mathbf{v}}_4 = (0,-1,0,-4)^T$. De forma matricial:

$$\underline{\underline{\mathbf{A}}}\, \underline{\underline{\mathbf{V}}} = \begin{pmatrix} 2 & 0 & -2 & 0 \\ 2 & -2 & 1 & -1 \\ 7 & -1 & -6 & 0 \\ -4 & 0 & 6 & -4 \end{pmatrix} \Leftrightarrow \underline{\underline{\mathbf{A}}} = \begin{pmatrix} 2 & 0 & -2 & 0 \\ 2 & -2 & 1 & -1 \\ 7 & -1 & -6 & 0 \\ -4 & 0 & 6 & -4 \end{pmatrix} \underline{\underline{\mathbf{V}}}^{-1} =$$

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 2 & -2 & 1 & -1 \\ 7 & -1 & -6 & 0 \\ -4 & 0 & 6 & -4 \end{pmatrix} \begin{pmatrix} -3 & 0 & 1 & 0 \\ 26 & 4 & -9 & -1 \\ 6 & 1 & -2 & 0 \\ 16 & 3 & -6 & -1 \end{pmatrix} = \begin{pmatrix} -18 & -2 & 6 & 0 \\ -68 & -10 & 24 & 3 \\ -83 & -10 & 28 & 1 \\ -16 & -6 & 8 & 4 \end{pmatrix}$$

(c) Procurando as imagens do $\underline{\mathbf{v}}_1, ..., \underline{\mathbf{v}}_4$:

$$f(\underline{\mathbf{v}}_1) = (1,1,0,0)^T]$$
$$f(\underline{\mathbf{v}}_2) = (-1,1,0,0)^T]$$
$$f(\underline{\mathbf{v}}_3) = (0,0,1,1)^T]$$
$$f(\underline{\mathbf{v}}_4) = (0,0,-1,1)^T]$$

Assim a matrix procurada é:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(d) Temos que: $dimU = 2$, pois $\underline{\mathbf{v}}_1$ e$\underline{\mathbf{v}}_2$ são linearmente independentes. $f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2 \in U$, e $f(\underline{\mathbf{v}}_2) = -\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2 \in U$, assim: $f(U) \subset U$. Porem as imagens $f(\underline{\mathbf{v}}_1), f(\underline{\mathbf{v}}_2)$ são linearmente independentes, assim: $dim f(U) = 2 = U$. Juntando: $f(U) = U$

(e) Obtivemos em item (a) a matriz inversa. Calculamos:

$$\underline{\underline{\mathbf{V}}}^{-1}\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{V}}} = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 26 & 4 & -9 & -1 \\ 6 & 1 & -2 & 0 \\ 16 & 3 & -6 & -1 \end{pmatrix} \begin{pmatrix} -18 & -2 & 6 & 0 \\ -68 & -10 & 24 & 3 \\ -83 & -10 & 28 & 1 \\ -16 & -6 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 0 & 1 & 0 \\ 4 & 3 & -3 & -3 \\ -2 & -2 & 5 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -3 & 0 & 1 & 0 \\ 26 & 4 & -9 & -1 \\ 6 & 1 & -2 & 0 \\ 16 & 3 & -6 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -2 & 0 \\ 2 & -2 & 1 & -1 \\ 7 & -1 & -6 & 0 \\ -4 & 0 & 6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \underline{\underline{\mathbf{B}}}$$

QED[1]

OBS! Respondendo a prova à lapis, perde-se o direito de revisão da prova. **OBS!**

[1] Quod Erat Demonstrandum: Seja provado!