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Curso:	Matemática
Disciplina:	Álgebra Linear
Prova:	II, 2^a chamada

1. *6 pts.*Dado os vetores:

$$\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{v}}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Mostre que os vetores $(\underline{\mathbf{v}}_i)$ formam uma base de $\mathbb{R}^4$.

(b) Encontrar uma relação expressando as coordenadas em relação a base canônica, em termo das coordenadas em relação a base $(\underline{\mathbf{v}}_i)$.

(c) Uma aplicação linear, $f : \mathbb{R}^4 \mapsto \mathbb{R}^3$, é dado por:

$$f(\underline{\mathbf{v}}_1) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad f(\underline{\mathbf{v}}_2) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad f(\underline{\mathbf{v}}_3) = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \quad f(\underline{\mathbf{v}}_4) = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$$

Encontrar o matriz, $\underline{\underline{\mathbf{A}}}'$, da f usando base canônica em $\mathbb{R}^3$ e base $(\underline{\mathbf{v}}_i)$ em $\mathbb{R}^4$.

(d) Encontrar o matriz, $\underline{\underline{\mathbf{A}}}$, da f usando base canônica em $\mathbb{R}^4$ e $\mathbb{R}^3$.

(e) Encontrar a dimensão da imagem, $f(\mathbb{R}^4)$.

(f) Dado os vetores: $\underline{\mathbf{d}}_1 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_3$ e $\underline{\mathbf{d}}_2 = -\underline{\mathbf{v}}_1 + 2\underline{\mathbf{v}}_2 - \underline{\mathbf{v}}_4$. Mostre que $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$ gera o núcleo da f.

(g) Encontrar, em termos de $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3, \underline{\mathbf{v}}_4$, todas as vetores que satisfaz: $f(\underline{\mathbf{x}}) = f(\underline{\mathbf{v}}_1)$.

Solution:

(a)

$$\left(\underline{\underline{\mathbf{V}}} \middle| \underline{\underline{\mathbf{I}}}\right) = \left(\begin{array}{cccc|cccc} \mathbf{1} & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & -1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

O que mostre que os vetores $\underline{\mathbf{v}}_i$ formam uma base em $\mathbb{R}^4$.

(b) Usando o cálculo no item anterior, temos:

$$\underline{\mathbf{x}}_{Antigo} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \underline{\mathbf{x}}_{Novo} \quad \Leftrightarrow \quad \underline{\mathbf{x}}_{Novo} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \underline{\mathbf{x}}_{Antigo}$$

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(c) Nas colunas da matriz, está as imagens dos vetores básicos:

$$\underline{\underline{\mathbf{A}}}' = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}$$

(d) Temos: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{V}}} = \underline{\underline{\mathbf{A}}}' \Leftrightarrow \underline{\underline{\mathbf{A}}} = \underline{\underline{\mathbf{A}}}'\,\underline{\underline{\mathbf{V}}}^{-1}$:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -8 & -1 & \frac{11}{2} & -\frac{3}{2} \\ 4 & -1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & -2 & \frac{7}{2} & -\frac{1}{2} \end{pmatrix}$$

(e)

$$\underline{\underline{\mathbf{A}}}' = \begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & -4 & -4 & 8 \\ 0 & -5 & -5 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & -4 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Assim: $\dim f(\mathbb{R}^4) = \rho_{\underline{\underline{\mathbf{A}}}'} = 2$.

(f) Primeiramente: $\dim \ker f = n - \rho = 4 - 2 = 2$. Observando que $\underline{\mathbf{d}}_1 = [1, 1, -1, 0]_{\underline{\mathbf{v}}}$ e $\underline{\mathbf{d}}_2 = [-1, 2, 0, 1]_{\underline{\mathbf{v}}}$:

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Ou seja: $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2 \in \ker f$. Por $\underline{\mathbf{d}}_1$ e $\underline{\mathbf{d}}_2$ sendo linearmente independentes, concluimos que eles geram o núcleo da f.

(g) Claro, que $\underline{\mathbf{x}} = \underline{\mathbf{v}}_1$ é uma solução particular da equação: $f(\underline{\mathbf{x}}) = f(\underline{\mathbf{v}}_1)$. Assim a solução completa desta equação é: $\underline{\mathbf{x}} = \underline{\mathbf{v}}_1 + \ker f = (1 + t - s)\underline{\mathbf{v}}_1 + (t + 2s)\underline{\mathbf{v}}_2 - t\underline{\mathbf{v}}_2 + s\underline{\mathbf{v}}_4, \quad t, s \in \mathbb{R}$.

2. *2 pts.* Sendo $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$ uma base em $\mathbb{C}^2$, uma aplicação linear, f, é dado por:

$$f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_1 + 2\underline{\mathbf{v}}_2 \quad f(\underline{\mathbf{v}}_2) = i\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$$

(a) Encontrar a matriz, $\underline{\underline{\mathbf{A}}}$, da f em relação a base $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$.

(b) Mostre que os vetores: $\underline{\mathbf{w}}_1 = \underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$ e $\underline{\mathbf{w}}_2 = \underline{\mathbf{v}}_1 - \underline{\mathbf{v}}_2$ formam uma base de $\mathbb{C}^2$.

(c) Encontrar a matriz, $\underline{\underline{\mathbf{B}}}$, da f em relação a base $\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2$.

Solution:

2 pts. Sendo $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2$ uma base em $\mathbb{C}^2$, uma aplicação linear, f, é dado por:

$$f(\underline{\mathbf{v}}_1) = \underline{\mathbf{v}}_1 + 2\underline{\mathbf{v}}_2 \quad f(\underline{\mathbf{v}}_2) = i\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2$$

(a) As imagens das vetores básicas são dadas, assim:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix}$$

(b)

$$(\underline{\underline{\mathbf{W}}}|\underline{\underline{\mathbf{I}}}) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \sim$$
$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right) = (\underline{\underline{\mathbf{I}}}|\underline{\underline{\mathbf{W}}}^{-1})$$

(c) A matrix na base nova, é: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{W}}}^{-1}\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{W}}} =$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+i & 1-i \\ 3 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4+i & 2-i \\ i-2 & -i \end{pmatrix}$$

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3. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 5 & 1 & -1 \\ -4 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

(a) Encontrar as imagens dos vetores: $\underline{\mathbf{v}}_1 = (1,-1,1)^T$ e $\underline{\mathbf{v}}_2 = (1,-2,2)^T$.

(b) Encontrar um vetor, $\underline{\mathbf{v}}_3$, tal que: $f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3$.

(c) Encontrar a matriz da f em relação a base $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3$.

Solution:

(a)

$$f(\underline{\mathbf{v}}_1) = \begin{pmatrix} 5 & 1 & -1 \\ -4 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 3\underline{\mathbf{v}}_1$$

$$f(\underline{\mathbf{v}}_2) = \begin{pmatrix} 5 & 1 & -1 \\ -4 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \underline{\mathbf{v}}_2$$

(b) $f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3 \Leftrightarrow \underline{\underline{\mathbf{A}}}\,\underline{\mathbf{v}}_3 = \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3 \Leftrightarrow (\underline{\underline{\mathbf{A}}} - \underline{\underline{\mathbf{I}}})\underline{\mathbf{v}}_3 = \underline{\mathbf{v}}_2$. Resolvendo este sistema:

$$\left(\begin{array}{rrr|r} 4 & 1 & -1 & 1 \\ -4 & 0 & 2 & -2 \\ 4 & 0 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{rrr|r} 4 & 1 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 0 & 1 & 1 & -1 \\ 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Usando z como parâmetro, $z = t$: $x = \frac{1}{2}(t+1)$ e $y = -t-1$. Ou seja, a solução completa é:

$$\underline{\mathbf{v}}_3 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

(c) Observando: $f(\underline{\mathbf{v}}_1) = 3\underline{\mathbf{v}}_1 = [3,0,0]_{\underline{\mathbf{v}}}$, $f(\underline{\mathbf{v}}_2) = \underline{\mathbf{v}}_2 = [0,1,0]_{\underline{\mathbf{v}}}$ e $f(\underline{\mathbf{v}}_3) = \underline{\mathbf{v}}_2 + \underline{\mathbf{v}}_3 = [0,1,1]_{\underline{\mathbf{v}}}$, obtemos a matriz da f colocando as imagens em colunas:

$$\underline{\underline{\mathbf{B}}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

OBS! Respondendo a prova à lapis, perde-se o direito de revisão da prova. **OBS!**