Dr. Ole Peter Smith

Instituto de Matemática e Estatística
Universidade Federal de Goiás
ole@mat.ufg.br - http://www.ime.ufg.br/docentes/olepeter

Data:	06/04/2010
Semestre:	2010.1
Curso:	Engenharia de Alimentos
Disciplina:	Álgebra Linear
Prova:	I

1. *2 pts.* Calcular os determinantes:

   (a)
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 4 & 1 & 2 \end{vmatrix}$$

   (b)
$$\begin{vmatrix} 0 & 3 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 7 \\ 1 & 0 & 0 & 3 \end{vmatrix}$$

Solution:

(a)
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 4 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -3 & -7 \\ 0 & -2 & -10 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -2 & -8 \end{vmatrix} =$$

$$\begin{vmatrix} -3 & -7 \\ -2 & -10 \end{vmatrix} - \begin{vmatrix} -3 & -4 \\ -2 & -8 \end{vmatrix} = 30 - 14 - (24 - 8) = 16 - 16 = 0$$

(b)
$$\begin{vmatrix} 0 & 3 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 7 \\ 1 & 0 & 0 & 3 \end{vmatrix} = (-1) \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 5 & 7 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 1 \\ -4 & 1 & 2 \\ -17 & -2 & 7 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & -7 & 3 \\ 2 & -11 & 5 \end{vmatrix} =$$

$$- \begin{vmatrix} -4 & 1 \\ -17 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -7 \\ 2 & -11 \end{vmatrix} = -(8 + 17) + 3(-11 + 14) = -25 + 9 = -16$$

2. *3 pts.* Dado a matriz e os vetores:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \quad \underline{\mathbf{b}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\mathbf{b}}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   (a) Mostre que a matriz $\underline{\underline{\mathbf{A}}}$ é regular.

   (b) Resolver o sistema $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}_1$.

   (c) Resolver o sistema $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}_2$.

Solution:

$$\left(\begin{array}{rrrrr|rr} 1 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & {}^*1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & {}^*1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|rr} 1 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|rr} 1 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & {}^*1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrrr|rr} 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & {}^*1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|rr} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|rr} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

(a) O cálculo anterior mostra que $\underline{\underline{\mathbf{A}}}$ é regular.

(b) Concluímos que a solução completa do sistema $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}_1$ é: $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

(c) Concluímos que a solução completa do sistema $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}_2$ é: $x_1 = 1 x_2 = x_3 = x_4 = x_5 = 0$.

3. *2 pts.* Considerando a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Encontrar $\det \underline{\underline{\mathbf{A}}}$.

(b) Encontrar a matriz inversa: $\underline{\underline{\mathbf{A}}}^{-1}$.

(c) Encontrar a matriz adjunta: $\underline{\underline{\mathbf{A}}}^*$.

Solution:

(a)

$$\det \underline{\underline{\mathbf{A}}} = \begin{vmatrix} 0 & {}^*1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & {}^*1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

(b) Por $\det \underline{\underline{\mathbf{A}}} \neq 0$, a inversa existe. Calculamos:

$$(\underline{\underline{\mathbf{A}}}|\,\underline{\underline{\mathbf{I}}}\,) = \left(\begin{array}{rrrr|rrrr} 0 & {}^*1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & {}^*1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{rrrr|rrrr} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right) = (\,\underline{\underline{\mathbf{I}}}\,|\underline{\underline{\mathbf{A}}}^{-1})$$

(c) Temos:

$$\underline{\underline{\mathbf{A}}}^* = (\det \underline{\underline{\mathbf{A}}})\underline{\underline{\mathbf{A}}}^{-1} = \underline{\underline{\mathbf{A}}}^{-1} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

4. *3 pts.* Dado as matrizes:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \quad \underline{\underline{\mathbf{B}}} = \begin{pmatrix} -2 & 5 \\ -2 & -4 \\ 3 & 0 \\ 1 & 2 \end{pmatrix}$$

(a) Encontrar o posto da $\underline{\underline{\mathbf{A}}}$.

(b) Resolver o sistema matricial: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{B}}}$.

(c) Indicar no item anterior a solução completa do sistema homogênea e uma solução particular do sistema não-homogênea.

Solution:

(a)

$$\underline{\underline{\mathbf{T}}} = (\underline{\underline{\mathbf{A}}}|\underline{\underline{\mathbf{B}}}) = \left(\begin{array}{cccc|cc} {}^{*}1 & 1 & 0 & -1 & -2 & 5 \\ 1 & 1 & 1 & -1 & -2 & -4 \\ -1 & -1 & 0 & 2 & 3 & 0 \\ 1 & 1 & 2 & 2 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 1 & 1 & 0 & -1 & -2 & 5 \\ 0 & 0 & {}^{*}1 & 0 & 0 & -9 \\ 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 3 & 3 & -3 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & -1 & -2 & 5 \\ 0 & 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 0 & {}^{*}1 & 1 & 5 \\ 0 & 0 & 0 & 3 & 3 & 15 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 1 & 1 & 0 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Vemos que: $\rho_A = \rho_T = 3$.

(b) Já que $\rho_A = \rho_T$, sabemos que o sistema matricial *tem* soluções, e mais: precisamos introduzir $n = \rho = 4 - 3 = 1$ parâmetro por coluna, para resolver-lo.

Resolvemos inicialmente o sistema homogênea: $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$. Temos: $x_1 + x_2 = 0 \wedge x_3 = x_3 = 0$. Pondo $x_2 = t$: $x_1 = -t \wedge x_3 = x_4 = 0$.

Isto é:

$$\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{0}} \Leftrightarrow \underline{\mathbf{x}} = t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

Similarmente para o sistema matricial homogênea: $\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{0}}}$ com duas colunas:

$$\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{0}}} \Leftrightarrow \underline{\underline{\mathbf{X}}} = \begin{pmatrix} -t & -s \\ t & s \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

Pela redução do item anterior, sabemos que uma solução particular do sistema não-homogênea é:

$$\underline{\underline{\mathbf{X}}}_p = \begin{pmatrix} -1 & 10 \\ 0 & -9 \\ 1 & 5 \\ 0 & 0 \end{pmatrix}$$

Adicionando a solução completa do sistema homogênea (SCSH) com a solução particular do sistema não-homogênea (SPSñH), obtemos finalmente a solução completa do sistema não-homogênea (SCSñH):

$$\underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{X}}} = \underline{\underline{\mathbf{B}}} \Leftrightarrow \underline{\underline{\mathbf{X}}} = \begin{pmatrix} -1 & 10 \\ 0 & -9 \\ 1 & 5 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -t & -s \\ t & s \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

(c) Repetindo comentários anteriores:

$$\underline{\underline{\mathbf{X}}}_p = \begin{pmatrix} -1 & 10 \\ 0 & -9 \\ 1 & 5 \\ 0 & 0 \end{pmatrix}$$

é uma solução particilar do sistema não-homogênea. E:

$$\underline{\underline{\mathbf{X}}}_H = \begin{pmatrix} -t & -s \\ t & s \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad t, s \in \mathbb{R}$$

é a solução completa do sistema homogênea.

OBS! Respondendo a prova à lapis, perde-se o direito de revisão da prova. **OBS!**