Dr. Ole Peter Smith
Instituto de Matemática e Estatística
Universidade Federal de Goiás
ole@mat.ufg.br

Data:	23/03/2010
Curso:	Matemática
Disciplina:	Álgebra Linear
Prova:	I

1. *2 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix}, \quad a \in \mathbb{R}$$

   (a) Encontrar o posto do $\underline{\underline{\mathbf{A}}}$ por todo $a \in \mathbb{R}$.

   (b) Resolver a equação: $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ por todo $a \in \mathbb{R}$.

   (c) Resolver a equaç ão: $\underline{\underline{\mathbf{A}}}^T\,\underline{\mathbf{x}} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$ por todo $b \in \mathbb{R}$.

   (d) Identifique a solução completa do sistema homogênea (SCSH) e uma solução particular do sistema não homogênea (SPSñH) no item anteriror.

Solution:

(a)

$$\begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & 1+a^2 \end{pmatrix}$$

Dali segue: $\rho_A = 2$ para todo $a \in \mathbb{R}$, pois

$$\begin{vmatrix} 1 & a \\ 0 & 1+a^2 \end{vmatrix} = 1 + a^2 \neq 0$$

(b) A sistema é equivalente à: $(a-1)y + (1+a^2)z = 0 \wedge x + y + az = 0$.
Para $a = 1$, obtemos: $z = 0 \wedge y = t \wedge x = -t$:

$$\underline{\mathbf{x}} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}$$

Para $a \neq 1$, obtemos: $z = t \wedge y = \frac{1+a^2}{1-a}t \wedge x = \frac{1+a^2}{1-a}t + at = \frac{1+a}{1-a}t$.

$$\underline{\mathbf{x}} = t \begin{pmatrix} \frac{1+a}{1-a} \\ \frac{1+a^2}{1-a} \\ 1 \end{pmatrix} = t' \begin{pmatrix} 1+a \\ 1+a^2 \\ 1-a \end{pmatrix}, \quad t' \in \mathbb{R}$$

(c)

$$\begin{pmatrix} 1 & -a & 0 \\ 1 & -1 & b \\ a & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -a & 0 \\ 0 & a-1 & b \\ 0 & 1+a^2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -a & 0 \\ 0 & a-1 & b \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 1 & 0 \end{pmatrix}$$

Vemos, que este sistem tem solução se e somente se: $b = 0$, e neste caso somente a solução trivial: $x = y = 0$.

(d) A solução particular no item anterior é a solução trivial, e a solução completa do sistema não homogênea também.

2. *2 pts.* Com o matriz no item anterior pomos:

$$\underline{\underline{\mathbf{B}}}_1 = \underline{\underline{\mathbf{A}}}^T \underline{\underline{\mathbf{A}}} \qquad \underline{\underline{\mathbf{B}}}_2 = \underline{\underline{\mathbf{A}}}\,\underline{\underline{\mathbf{A}}}^T$$

   (a) Calcular $\underline{\underline{\mathbf{B}}}_1$ e $\underline{\underline{\mathbf{B}}}_2$.

(b) Calcular os determiantes det $\underline{\underline{\mathbf{B}}}_1$ e det $\underline{\underline{\mathbf{B}}}_2$.

(c) Calcular as adjuntas $\underline{\underline{\mathbf{B}}}_1^*$ e $\underline{\underline{\mathbf{B}}}_2^*$.

(d) Justifique que o produto de uma matriz com sua transposta é uma matriz simétrica.

Solution:

(a)

$$\underline{\underline{\mathbf{B}}}_1 = \begin{pmatrix} 1 & -a \\ 1 & -1 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+a^2 & 1+a & 0 \\ 1+a & 2 & a-1 \\ 0 & a-1 & 1+a^2 \end{pmatrix}$$

$$\underline{\underline{\mathbf{B}}}_2 = \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 1 & -1 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 2+a^2 & -1 \\ -1 & 2+a^2 \end{pmatrix}$$

(b)

$$\det \underline{\underline{\mathbf{B}}}_1 = \begin{vmatrix} 1+a^2 & 1+a & 0 \\ 1+a & 2 & a-1 \\ 0 & a-1 & 1+a^2 \end{vmatrix} = (1+a^2) \begin{vmatrix} 2 & a-1 \\ a-1 & 1+a^2 \end{vmatrix} - (1+a) \begin{vmatrix} 1+a & 0 \\ a-1 & 1+a^2 \end{vmatrix} =$$

$$(1+a^2)\left[2+2a^2-(a-1)^2\right] - (1+a)\left[(1+a)(1+a^2)\right] =$$
$$(1+a^2)\left[2+2a^2-a^2+2a-1\right] - (1+a)^2(1+a^2) =$$
$$(1+a^2)\left[a^2+2a+1-(a+1)^2\right] = 0$$

$$\det \underline{\underline{\mathbf{B}}}_2 = \begin{vmatrix} 2+a^2 & -1 \\ -1 & 2+a^2 \end{vmatrix} = (2+a^2)^2 - 1 = a^4 + 4a^2 + 3$$

(c) Por $\underline{\underline{\mathbf{B}}}_1$ ser simétrica, a adjunta também é:

$$C_{11} = A_{11} = 2(a^2-1) - (a-1)^2 = 2a^2 - 2 - a^2 + 2a - 1 = a^2 + 2a - 3$$
$$C_{12} = -A_{12} = (a^2+1)(1+a)$$
$$C_{13} = A_{13} = (1+a)(a-1) = a^2 - 1$$
$$C_{22} = A_{22} = (a^2+1)^2$$
$$C_{23} = -A_{23} = (a^2+1)(a-1)$$
$$C_{33} = A_{33} = 2(a^2+1) - (a+1)^2 = a^2 - 2a$$

Juntando:

$$\underline{\underline{\mathbf{B}}}_1^* = \begin{pmatrix} a^2+2a-3 & (a^2+1)(1+a) & a^2-1 \\ (a^2+1)(1+a) & (a^2+1)^2 & (a^2+1)(a-1) \\ a^2-1 & (a^2+1)(a-1) & a^2-2a \end{pmatrix}$$

Mesmo por $\underline{\underline{\mathbf{B}}}_1$:

$$C_{11} = A_{11} = 2 + a^2$$
$$C_{12} = -A_{12} = -1$$
$$C_{22} = A_{22} = 2 + a^2$$

E juntando:

$$\underline{\underline{\mathbf{B}}}_1^* = \begin{pmatrix} 2+a^2 & -1 \\ -1 & 2+a^2 \end{pmatrix}$$

(d) $(\underline{\underline{\mathbf{A}}}^T \underline{\underline{\mathbf{A}}})^T = \underline{\underline{\mathbf{A}}}^T (\underline{\underline{\mathbf{A}}}^T)^T = \underline{\underline{\mathbf{A}}}^T \underline{\underline{\mathbf{A}}}$.

3. *3 pts.*

   (a) Mostre que o determinante de ordem $n > 1$:

$$A_n = \begin{vmatrix} b & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ a & b & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & a & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & \dots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & b \end{vmatrix}$$

   satisfaz a fórmula de recursão: $A_n = bA_{n-1} - a^2 A_{n-2}, \quad n \geq 3$.

   (b) Pondo $A_1 = b$, encontrar A_3.

   (c) Encontrar a determinante:

$$A = \begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & -1 & 2-\lambda \end{vmatrix}$$

Solution:

   (a) Desenvolvimento de Laplace pela primeira linha:

$$A_n = \begin{vmatrix} b & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ a & b & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & a & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & \dots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & b \end{vmatrix} =$$

$$b \begin{vmatrix} b & a & 0 & \dots & 0 & 0 & 0 & 0 \\ a & b & a & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 & 0 & 0 \\ & \vdots & & \ddots & & & \vdots & \\ 0 & 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & 0 & \dots & 0 & 0 & a & b \end{vmatrix} - a \begin{vmatrix} a & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & a & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 & 0 & 0 \\ & \vdots & & \ddots & & & \vdots & \\ 0 & 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & 0 & \dots & 0 & 0 & a & b \end{vmatrix}$$

$$= bA_{n-1} - a^2 \begin{vmatrix} a & 0 & \dots & 0 & 0 & 0 & 0 \\ b & a & \dots & 0 & 0 & 0 & 0 \\ a & b & \dots & 0 & 0 & 0 & 0 \\ & \vdots & \ddots & & & \vdots & \\ 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & \dots & 0 & 0 & a & b \end{vmatrix} = bA_{n-1} - a^2 A_{n-2}$$

   QED.

(b) $A_1 = b$ e $A_2 = b^2 - a^2$. Assim:
$A_3 = b(b^2 - a^2) - a^2 b = b^3 - 2a^2 b$.

(c) Similarmente:
$A_4 = bA_3 - a^2 A_2 = b^2(b^2 - 2a^2) - a^2(b^2 - a^2) = b^4 - 2a^2b^2 - a^2b^2 + a^4 = b^4 - 3a^2b^2 + a^4$. Pondo $b = 2 - \lambda$ e $a = -1$:
Encontrar a determinante:

$$A = \begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^4 - 3(2-\lambda)^2 + 1$$

4. *3 pts.* Dado a matriz:

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} \alpha & \alpha & 0 & 1 \\ 2\alpha & 2\alpha & \alpha & 1 \\ 2\alpha & 3\alpha & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

(a) Encontrar o posto do $\underline{\underline{\mathbf{A}}}$ por todo $\alpha \in \mathbb{R}$.

(b) Por quais valores $\alpha \in \mathbb{R}$ $\underline{\underline{\mathbf{A}}}$ é regular? Por estes valores, encontrar a inversa.

(c) Resolver a sistema:

$$\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ a \\ b \\ 0 \end{pmatrix}$$

por todos $\alpha, a, b \in \mathbb{R}$.

Solution:

(a)

$$\begin{pmatrix} \alpha & \alpha & 0 & 1 \\ 2\alpha & 2\alpha & \alpha & 1 \\ 2\alpha & 3\alpha & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 1 \\ 0 & \alpha & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Vemos que o posto é 2 para $\alpha = 0$ e 4 para $\alpha \neq 0$.

(b) Pelo resultado no item anterior, $\underline{\underline{\mathbf{A}}}$ tem inversa se e somente se: $\alpha \neq 0$. Encontramos a inversa:

$$\left(\begin{array}{cccc|cccc} \alpha & \alpha & 0 & 1 & 1 & 0 & 0 & 0 \\ 2\alpha & 2\alpha & \alpha & 1 & 0 & 1 & 0 & 0 \\ 2\alpha & 3\alpha & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 1 & 0 & 1 & 0 & -2\alpha \\ 0 & \alpha & 0 & 1 & 0 & 0 & 1 & -2\alpha \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 & -1 & 1 & 0 & -\alpha \\ 0 & \alpha & 0 & 0 & -1 & 0 & 1 & -\alpha \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 & -\alpha^{-1} & \alpha^{-1} & 0 & -1 \\ 0 & 1 & 0 & 0 & -\alpha^{-1} & 0 & \alpha^{-1} & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 & -\alpha^{-1} & \alpha^{-1} & 0 & -1 \\ 0 & 1 & 0 & 0 & -\alpha^{-1} & 0 & \alpha^{-1} & -1 \\ 1 & 0 & 0 & 0 & \alpha^{-1} & 0 & -\alpha^{-1} & 2 \end{array}\right)$$

Trocando a ordem das linhas obtemos a inversa:

$$\begin{pmatrix} \alpha^{-1} & 0 & -\alpha^{-1} & 2 \\ -\alpha^{-1} & 0 & \alpha^{-1} & -1 \\ -\alpha^{-1} & \alpha^{-1} & 0 & -1 \\ 1 & 0 & 0 & -\alpha \end{pmatrix}$$

(c) Quanto a inversa existe ($\alpha \neq 0$), obtemos:

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{A}}}^{-1} \begin{pmatrix} 1 \\ a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 & -\alpha^{-1} & 2 \\ -\alpha^{-1} & 0 & \alpha^{-1} & -1 \\ -\alpha^{-1} & \alpha^{-1} & 0 & -1 \\ 1 & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} (1-b)\alpha^{-1} \\ (-1+b)\alpha^{-1} \\ (-1+a)\alpha^{-1} \\ 1 \end{pmatrix}$$

Para $\alpha = 0$ obtemos:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & b \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & a-1 \\ 0 & 0 & 0 & 0 & b-1 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Temos solução se e somente se: $a = b = 1$, e neste caso:

$$\underline{\mathbf{x}} = \begin{pmatrix} t \\ -t \\ s \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R}$$